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2022 VCAA Mathematical Methods Exam 2 Solutions

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SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	A	E	D	C	C	E	E	D	D

11	12	13	14	15	16	17	18	19	20
C	A	C	B	E	B	A	B	D	A

Q1 Period $=\frac{2\pi}{2}=\pi$ B

Q2 As $x\to\infty$, $y\to 4$ A

Q3 When $x=0$, $\frac{dy}{dx}=3e^{3x}=3$ E

Q4 $f(x)=\tan\frac{x}{3}$ is undefined at $\frac{x}{3}=\frac{\pi}{2}$, i.e. $x=\frac{3\pi}{2}<5$ D

Q5 $f'(x)=2x+3=0$ when $x=-\frac{3}{2}$, i.e. turning point at $x=-\frac{3}{2}$

$\therefore\ a=-1.5$ C

Q6 Inverse of $y=x^2$ for $x<0$ is $x=y^2$ for $y<0$

i.e. $y=-\sqrt{x}$ C

Q7 Locate the points of zero gradient, after the second point the gradient is positive. E

Q8 $\int_0^b f(x)dx=\int_0^a f(x)dx+\int_a^b f(x)dx$, $10=-4+\int_a^b f(x)dx$

$\therefore\ \int_a^b f(x)dx=14$ E

Q9 Shortest distance is straight line distance.

$d=\sqrt{x^2+y^2}=\sqrt{x^2+2x+1}=\sqrt{(x+1)^2}=x+1$ D

Q10 $n=1000$, $\hat{p}=0.55$, approx 95% confidence interval for $\hat{P}$

is $\left(0.55-1.96\sqrt{\frac{0.55\times 0.45}{1000}},\ 0.55+1.96\sqrt{\frac{0.55\times 0.45}{1000}}\right)$

$\approx(0.519,\ 0.581)$ or $(51.9\%,\ 58.1\%)$ D

Q11 $x\cos x=\frac{d}{dx}(x\sin x)-\sin x$,

$\frac{1}{k}\int x\cos x dx=\frac{1}{k}\left(\int\frac{d}{dx}(x\sin x)dx-\int\sin x dx\right)$

$=\frac{1}{k}\left(x\sin x-\int\sin x dx\right)+c$ C

Q12 $\Pr(RB)+\Pr(BR)=\frac{3}{3+x}\times\frac{x}{2+x}+\frac{x}{3+x}\times\frac{3}{2+x}=\frac{6x}{(3+x)(2+x)}$

A

Q13 $\frac{x+a}{x-a}>0$,

either both $x+a$ and $x-a$ are both positive, i.e. $x>a$

or both are negative, i.e. $x<-a$ $\therefore\ x\in R\setminus[-a,\ a]$ C

Q14 $\int_0^{100}\frac{2}{9}x^2 e^{-\frac{1}{9}x^2}dx\approx 2.659$ B

Q15 $x^2-2x-3\ge 0$, $(x+1)(x-3)\ge 0$, E

Q16 Solve $f'(-3)=9-6m+n=0$, $f'(1)=9+2m+n=0$

$\therefore\ m=1$ and $n=-3$, $\therefore\ f(x)=\frac{1}{3}x^3+x^2-3x+p$, $(3,\ 4)$ $\therefore\ p=-5$ B

Q17 Property 1 gives $(a,\cdots)$ is below $(b,\cdots)$. Property 2 shows $g(x)$ is a cubic shape function with turning points. A

Q18 $\Pr(X\ge 16\mid X\ge a)=0.9175$ and $a\le 15$

$\frac{\Pr(X\ge 16\cap X\ge a)}{\Pr(X\ge a)}=\frac{\Pr(X\ge 16)}{\Pr(X\ge a)}=\frac{0.91728}{\Pr(X\ge a)}=0.9175$

$\therefore\ \Pr(X\ge a)\approx 0.99976$, $a=12$ B

Q19 $V=x(a-2x)(b-2x)=abx-2(a+b)x^2+4x^3$

$\frac{dV}{dx}=ab-4(a+b)x+12x^2=0$, $x=\frac{a+b-\sqrt{a^2-ab+b^2}}{6}$ D

Q20 $50\sin 2\theta>40$, $26.565<\theta<63.435$,

$\Pr(26.565<\theta<63.435)\approx 0.96947$ A

SECTION B

Q1a $x=0$

Q1b $f'(x)=\frac{1}{6}x$

Q1c $f'(x)=\frac{1}{6}x=-2$, $x=-12$, $y=f(-12)=12$

Tangent at M: $y-12=-2(x-(-12))$ $\therefore\ y=-2x-12$

Q1di Perpendicular at M: $y-12=\frac{1}{2}(x-(-12))$, $y=\frac{1}{2}x+18$

Q1dii Point N: $y=\frac{1}{2}x+18$ and $y=\frac{x^2}{12}$ $\therefore\ x=18$

$\int_{-12}^{18}\left(\frac{1}{2}x+18-\frac{x^2}{12}\right)dx=375$

Q1e All parabolas are similar. To simplify calculations, consider $y=x^2$. $M(-p,\ p^2)$ and the perpendicular at M is $y=\frac{1}{2p}x+p^2+\frac{1}{2}$ which intersects $y=x^2$ at $x=p+\frac{1}{2p}$.

Enclosed area $A=\int_{-p}^{p+\frac{1}{2p}}\frac{1}{2p}x+p^2+\frac{1}{2}-x^2\ dx$, $A_{\min}$ when $p=\frac{1}{2}$ and

$\therefore\ M\left(-\frac{1}{2},\frac{1}{4}\right)$. For $y=\frac{x^2}{4a^2}$, $M\left(-b,\frac{b^2}{4a^2}\right)$. $\because$ parabolas are similar,

$\therefore$ same dilation in both directions $\therefore\ -\frac{1}{2}:-b=\frac{1}{4}:\frac{b^2}{4a^2}$ $\therefore\ b=2a^2$

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Q2ai $r(0)=2500$

Q2aii Max $=1700+2500=4200$, min $=-1700+2500=800$

Q2aiii $200-40=160$

Q2b $(20, 700)$ and $(100, 2500)$, $a=\dfrac{2500-700}{2}=900$

Same period as rabbit population .: $b=\dfrac{\pi}{80}$

Q2c By CAS , max value of $r(t)+f(t)\approx 5339$

Q2d By CAS, 160

Q2e

For fox population, $\begin{bmatrix} t' \\ y' \end{bmatrix}=\begin{bmatrix} \frac{90}{\pi}t+60 \\ 900y+1600 \end{bmatrix}$, $\begin{bmatrix} t \\ y \end{bmatrix}=\begin{bmatrix} \frac{\pi}{90}(t'-60) \\ \frac{1}{900}(y'-1600) \end{bmatrix}$

$\dfrac{1}{900}(y'-1600)=\sin\dfrac{\pi}{90}(t'-60)$

.: $f(t)=900\sin\dfrac{\pi}{90}(t-60)+1600$

Average combined population $=\dfrac{\int_0^{300}(f(t)+r(t))dt}{300}\approx 4142$ by CAS

Q2f By CAS, the first two maxima are $(38.06, 4012.2)$ and $(198.06, 3435.7)$

.: average rate of change $\approx\dfrac{3435.7-4012.2}{198.06-38.06}\approx -3.6$

Q2g By CAS, $t\approx 156$

Q2h As $t\to\infty$, $s\to 2500$

Q3ai $n=5$, $p=\dfrac{1}{2}$, $\Pr(X=5)=\left(\dfrac{1}{2}\right)^5=\dfrac{1}{32}$

Q3aii $\Pr(X\ge 2)=0.8125=\dfrac{13}{16}$

Q3aiii $\Pr(X\ge 2 \mid X<5)=\dfrac{\Pr(2\le X\le 4)}{\Pr(X\le 4)}\approx 0.806$

Q3aiv $\mathrm{E}(X)=np=\dfrac{5}{2}$, $\mathrm{sd}(X)=\sqrt{np(1-p)}=\sqrt{1.25}=\dfrac{\sqrt{5}}{2}$

Q3bi 1

Q3bii $\int_{1.5}^{2}(ah^2+bh+c)dh=\left[\dfrac{ah^3}{3}+\dfrac{bh^2}{2}+ch\right]_{1.5}^{2}=0.35$

$\int_{2}^{3}(ah^2+bh+c)dh=\left[\dfrac{ah^3}{3}+\dfrac{bh^2}{2}+ch\right]_{2}^{3}=0.25$

$\int_{1.5}^{3}(ah^2+bh+c)dh=\left[\dfrac{ah^3}{3}+\dfrac{bh^2}{2}+ch\right]_{2}^{3}=1$

Solve simultaneous equations by CAS, $a=-\dfrac{4}{5}$, $b=\dfrac{17}{5}$, $c=-\dfrac{167}{60}$

Q3biii $h+d=3$.: $f(h)=f(3-d)=g(d)$.: $r=-1$ and $s=3$

Q3ci Since x and N are both discrete .: $\hat{P}=\dfrac{x}{N}$ is discrete.

Q3cii $\left(0.4-1.96\sqrt{\dfrac{0.4\times 0.6}{25}},\ 0.4-1.96\sqrt{\dfrac{0.4\times 0.6}{25}}\right)\approx(0.208, 0.592)$

Q3ciii By halving the standard deviation: $\dfrac{1}{2}\sqrt{\dfrac{0.4\times 0.6}{25}}=\sqrt{\dfrac{0.4\times 0.6}{100}}$

.: $n=100$

Q4a R is the range.

Q4bi $f'(x)=\dfrac{1}{x+\frac{1}{2}}+\dfrac{1}{\frac{1}{2}-x}$, $f'(0)=4$

Q4bii $\left(-\dfrac{1}{2},\dfrac{1}{2}\right)$

Q4c $f(x)+f(-x)$

$=\log_e\left(x+\dfrac{1}{2}\right)-\log_e\left(\dfrac{1}{2}-x\right)+\log_e\left((-x)+\dfrac{1}{2}\right)-\log_e\left(\dfrac{1}{2}-(-x)\right)=0$

Q4d Domain of f^{-1} is the range of f , i.e. R .

Rule: $x=\log_e\left(y+\dfrac{1}{2}\right)-\log_e\left(\dfrac{1}{2}-y\right)$, $x=\log_e\left(\dfrac{y+\frac{1}{2}}{\frac{1}{2}-y}\right)$, $\dfrac{y+\frac{1}{2}}{\frac{1}{2}-y}=e^x$

.: $y=f^{-1}(x)=\dfrac{e^x-1}{2(e^x+1)}$, $x\in R$

Q4ei $\dfrac{1}{k}$ is the vertical dilation factor of $f(x)$. For $A(k)>0$, $h'(0)<1$, $\dfrac{1}{k}(4)<1$.: $k>4$

Q4eii $k\ne 0$ for $h(x)$ to be defined, also for $A(k)\ge 0$, $k\ge 4$

Q5a $g\left(\dfrac{\pi}{6}\right)=f\left(\sin\dfrac{\pi}{3}\right)=f\left(\dfrac{\sqrt{3}}{2}\right)=3$

Q5b $g'\left(\dfrac{\pi}{6}\right)=2\cos\dfrac{\pi}{3}f'\left(\sin\dfrac{\pi}{3}\right)=2\times\dfrac{1}{2}f'\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{1}{9}$

Q5c Tangent at $\left(\dfrac{\pi}{6},3\right)$: $m=g'\left(\dfrac{\pi}{6}\right)=\dfrac{1}{9}$, $y-3=\dfrac{1}{9}\left(x-\dfrac{\pi}{6}\right)$

.: $y=\dfrac{1}{9}x+3-\dfrac{\pi}{54}$

Q5d Av of $g'(x)$ in the interval $=\dfrac{\int_{\frac{\pi}{8}}^{\frac{\pi}{6}}g'(x)dx}{\frac{\pi}{6}-\frac{\pi}{8}}=\dfrac{[g(x)]_{\frac{\pi}{8}}^{\frac{\pi}{6}}}{\frac{\pi}{24}}=-\dfrac{48}{\pi}$

Q5e $g'(x)=0$ and $0\le x\le\pi$

$\cos 2x=0$ or $f'(\sin 2x)=0$, i.e. $\sin 2x=\dfrac{\sqrt{2}}{2}$ from given table

.: $2x=\dfrac{\pi}{2},\dfrac{3\pi}{2}$ or $2x=\dfrac{\pi}{4},\dfrac{3\pi}{4}$.: $x=\dfrac{\pi}{8},\dfrac{\pi}{4},\dfrac{3\pi}{8},\dfrac{3\pi}{4}$