2022 VCAA Specialist Mathematics Exam 1 Solutions

Q1a $p(z) = (z+ai)^2 + b = z^2 + 2aiz - a^2 + b = z^2 + 6iz - 25$

$\therefore a = 3$ and $b = -16$ $\therefore p(z) = (z+3i)^2 - 16$

Q1b $p(z) = (z+3i)^2 - 16 = 0$ $\therefore (z+3i)^2 = 16$, $z = \pm 4 - 3i$

Q2 $\frac{dy}{dx} = -x\sqrt{4-y^2}$ and $y(2) = 0$; $\int \frac{-1}{\sqrt{4-y^2}}dy = \int x dx$

$\therefore \cos^{-1}\left(\frac{y}{2}\right) = \frac{x^2}{2} + c$ and $y(2) = 0$ $\therefore c = \frac{\pi}{2} - 2$,

$y = 2\cos\left(\frac{x^2}{2} + \frac{\pi}{2} - 2\right)$

Q3a For 1 cup, $\mu = 10$, $\sigma = 1.5$, $\text{Var}(X) = 1.5^2 = 2.25$

For 4 cups, $\mu = 4\times 10 = 40$, $\text{Var}(X) = 4\times 2.25 = 9$, $\sigma = \sqrt{9} = 3$

$34 = 40 - 2\times 3 = \mu - 2\sigma$ $\therefore \Pr(X > 34) = 1 - \frac{0.05}{2} = 9.75 \approx 9.8$

Q3b Note: Misleading statement "Following the service, the mean time taken to dispense 25 cups of coffee is found to be 9 s"?

$n = 25$, $\bar{x} = 9$, $\sigma = 1.5$

95% confidence interval for μ is

$\left(9 - 1.96\times\frac{1.5}{\sqrt{25}},\ 9 - 1.96\times\frac{1.5}{\sqrt{25}}\right) \approx (8.4, 9.6)$

Q4 $\frac{3x^2+4x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$, A, B and C are found to be 3, 0 and 4 respectively. $\int \frac{3x^2+4x+12}{x(x^2+4)}dx = \int \frac{3}{x} + \frac{4}{x^2+4}dx$

$= 3\ln|x| + 2\tan^{-1}\left(\frac{x}{2}\right) + c$

Q5a $t = 0$, $v = 0$, $\tan\theta = \frac{1}{3}$ $\therefore \sin\theta = \frac{1}{\sqrt{10}}$, $\cos\theta = \frac{3}{\sqrt{10}}$

$a = \frac{10g\sin\theta}{10} = \frac{g}{\sqrt{10}}$ constant acceleration $\therefore v = at = \frac{2g}{\sqrt{10}}$ m s^{-1}

Q5b $R = 10g\sin\theta = \sqrt{10}g$

Q6a $\tilde{a}.\tilde{b} = ab\cos\theta$, $8 = 21\cos\theta$, $\cos\theta = \frac{8}{21}$ and $0 < \theta < \frac{\pi}{2}$

Q6bi $\overrightarrow{OP} = x\tilde{i} + y\tilde{j}$, $\overrightarrow{OQ} = \overrightarrow{OP} - 2a\tilde{i} = (x-2a)\tilde{i} + y\tilde{j}$

Q6bii $\overrightarrow{OP}.\overrightarrow{QP} = x(x-2a) + y^2 = x(x-2a) + a^2 - (x-a)^2 = 0$

$\therefore \overrightarrow{OP} \perp \overrightarrow{QP}$

Q7 By implicit differentiation: $\frac{d}{dx}x\cos(x+y) = 0$

$x\left(-\sin(x+y)\times\left(1 + \frac{dy}{dx}\right)\right) + \cos(x+y) = 0$

$x\sin(x+y)\frac{dy}{dx} = \cos(x+y) - x\sin(x+y)$ $\therefore \frac{dy}{dx} = \frac{1}{x\tan(x+y)} - 1$

At $\left(\frac{\pi}{24}, \frac{7\pi}{24}\right)$, $\frac{dy}{dx} = \frac{1}{\frac{\pi}{24}\tan\left(\frac{\pi}{24} + \frac{7\pi}{24}\right)} - 1 = \frac{1}{\frac{\pi}{24}\tan\frac{\pi}{3}} - 1 = \frac{8\sqrt{3} - \pi}{\pi}$

Q8 $a = -4x$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$, $\frac{1}{2}v^2 = -2x^2 + c$

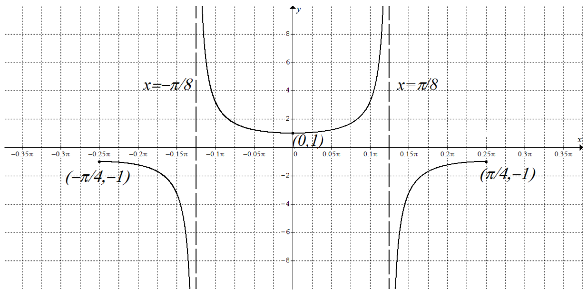
$v = -2$ at $x = 0$ $\therefore \frac{1}{2}v^2 = -2x^2 + 2$, $v^2 = 4(1-x^2)$, $v = -2\sqrt{1-x^2}$

Q9 $u = \sin 2x$, $\frac{1}{2}\frac{du}{dx} = \cos 2x$

$f(x) = \int \frac{\cos 2x}{\sin^3 2x}dx = \int \frac{1}{2}u^{-3}du = \frac{1}{-4u^2} + c$ and given $f\left(\frac{\pi}{8}\right) = \frac{3}{4}$

$\therefore c = \frac{5}{4}$ $\therefore f(x) = \frac{5}{4} - \frac{1}{4\sin^2 2x}$

Q10a



Q10b $V = \int_{-\frac{\pi}{24}}^{\frac{\pi}{48}} \pi\sec^2(4x)dx = \left[\frac{\pi}{4}\tan(4x)\right]_{-\frac{\pi}{24}}^{\frac{\pi}{48}}$

$= \frac{\pi}{4}\left(\tan\left(\frac{\pi}{12}\right) - \tan\left(-\frac{\pi}{6}\right)\right) = \frac{\pi}{4}\left(\tan\left(\frac{\pi}{12}\right) + \tan\left(\frac{\pi}{6}\right)\right)$

Note: $\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1 - \tan^2\frac{\pi}{12}}$, let $x = \tan\frac{\pi}{12} > 0$ $\therefore \frac{1}{\sqrt{3}} = \frac{2x}{1-x^2}$

Solve to obtain $x = \tan\frac{\pi}{12} = 2 - \sqrt{3}$

$\therefore V = \frac{\pi}{4}\left(2 - \sqrt{3} + \frac{1}{\sqrt{3}}\right) = \frac{\pi}{4}\left(\frac{2\sqrt{3} - 2}{\sqrt{3}}\right) = \frac{(3-\sqrt{3})\pi}{6}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.