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# 2022 VCAA Specialist Mathematics Exam 2 Solutions



**Use CAS to save time**

**SECTION A – Multiple-choice questions**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| B | E | E | B | A | E | D | C | B | E |

| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| A | A | B | D | E | A | C | B | D | C |

Q1 By CAS **B**

Q2 $1-\dfrac{4\sin^2 x}{\tan^2 x+1}=1-\dfrac{4\sin^2 x}{\sec^2 x}=1-(2\sin x\cos x)^2$
$=1-\sin^2 2x=\cos^2 2x$ **E**

Q3 Consider $c=0$ which has 1 horizontal and 1 vertical asymptotes. **E**

Q4 $b$ is real .: $a$ and $c$ are complex conjugates assuming that $\text{Im}(a)\neq 0$ and $\text{Im}(c)\neq 0$ .: $|a|=|c|$ **B**

Q5 $\dfrac{y-1}{x}=\tan\dfrac{3\pi}{4}$ .: $y=1-x$ and $x<0$ **A**

Q6 Circle $|z-5-5i|=4$ intersects $|z-5|=2$ at two points **E**

Q7 $u=1+e^x$, $\dfrac{du}{dx}=e^x$, $\dfrac{1}{e^x}\dfrac{du}{dx}=1$, $\dfrac{1}{u-1}\dfrac{du}{dx}$

$\displaystyle\int_0^{\log_e 2}\frac{1}{u}.\frac{1}{u-1}\frac{du}{dx}dx=\int_2^3\frac{1}{u(u-1)}du=\int_2^3\left(\frac{1}{u-1}-\frac{1}{u}\right)du$ **D**

Q8 $x=0$, $\dfrac{dy}{dx}=0$ .: either A, B or C

When $x<0$ and $y>0$, $\dfrac{dy}{dx}>0$ .: not A

At $(-1,1)$, $\dfrac{dy}{dx}>1$ .: C **C**

Q9 $x_0=1$ $y_0=2$ $\dfrac{dy}{dx}=2x^2=2$

$x_1=1+h$, $y_1=2+2h$ $\dfrac{dy}{dx}=2(1+h)^2$

$x_2=1+2h$ $y_2=2+2h+h\times 2(1+h)^2=2.976$ .: $h=0.2$ **B**

Q10 $5x^2y-3xy+y^2=10$ At $(1,m)$, $5m-3m+m^2=10$
.: $m=-1\pm\sqrt{11}$. By checking the graph both values give negative gradient. **E**

Q11 Let $\tilde{c}=\alpha\tilde{a}+\beta\tilde{b}$,
$-3\tilde{i}+2\tilde{j}+5\tilde{k}=(2\alpha+\beta)\tilde{i}+(-3\alpha+2\beta)\tilde{j}+(\alpha p-\beta q)\tilde{k}$
Equate coefficients: $2\alpha+\beta=-3$, $-3\alpha+2\beta=2$, $\alpha p-\beta q=5$
Solve the first two to obtain $\alpha=-\dfrac{8}{7}$ and $\beta=-\dfrac{5}{7}$
Substitute into the third equation to obtain $8p=5q-35$ **A**

Q12 $\tilde{u}.\tilde{v}=0$, $-\text{cosec}x\cos x+\sqrt{3}=0$, $-\dfrac{\cos x}{\sin x}+\sqrt{3}=0$,
$\tan x=\dfrac{1}{\sqrt{3}}$, $x=\dfrac{\pi}{6},\dfrac{7\pi}{6}$ **A**

Q13 $\ddot{r}=\sin t\,\tilde{i}+2\cos t\,\tilde{j}$, $\dot{r}=-\cos t\,\tilde{i}+2\sin t\,\tilde{j}+\tilde{c}$
$\dot{r}(0)=2\tilde{i}+\tilde{j}=-\tilde{i}+\tilde{c}$ .: $\tilde{c}=3\tilde{i}+\tilde{j}$
.: $\dot{r}=(3-\cos t)\tilde{i}+(1+2\sin t)\tilde{j}$ **B**

Q14 $17^2=7^2+2as$ .: $s=\dfrac{120}{a}$, $\dfrac{1}{2}s=\dfrac{60}{a}$, $v^2=7^2+2a\left(\dfrac{60}{a}\right)$ and $v$ is in the same direction as 7 and 17 .: $v=\sqrt{169}=13$ **D**

Q15 Newton's second law. **E**

Q16 The three forces form a closed triangle when they are put (added) head to tail. $10^2=5^2+7^2-2\times5\times7\cos(\pi-\theta)$
.: $100=25+49+70\cos\theta$ **A**

Q17 $s=\dfrac{1}{2}(u+v)t$, $30=\dfrac{1}{2}(3+v)6$ .: $v=7$
.: $\Delta p=m\times(v-u)=7(7-3)=28$ **C**

Q18 For $T$, $\mu=30$, $\text{Var}(T)=2.5^2=6.25$. For $T_d=T_1-T_2$,
$\text{E}(T_d)=30-30=0$, $\text{Var}(T_d)=6.25+6.25=12.5$ .: $\text{sd}(T_d)=\sqrt{12.5}$
$\Pr(T_d<-6\cup T_d>6)=1-\Pr(-6<T_d<6)\approx 0.0897$ **B**

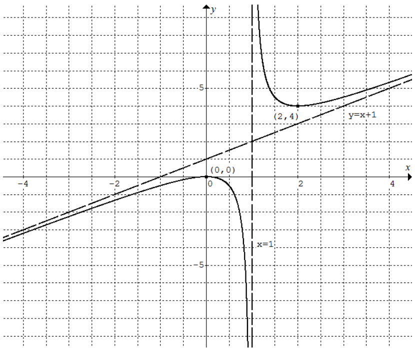
Q19 $\mu=\text{E}(C)=\text{E}(0.3m+0.5)=0.3\text{E}(m)+0.5=2.6$
$\text{Var}(C)=0.3^2\,\text{Var}(m)=0.0009$, $\sigma=\text{sd}(C)=\sqrt{0.0009}=0.03$
Let $\overline{X}$ be the sample mean random variable. For a sample of 100,
$\text{E}(\overline{X})=\mu=2.6$, $\text{sd}(\overline{X})=\dfrac{0.03}{\sqrt{100}}=0.003$
The question asked for $a$ such that $\Pr(2.6-a<\overline{X}<2.6+a)=0.95$
.: $a\approx 0.05880$ and the 95% confidence interval $\approx(2.594, 2.606)$ **D**

Q20 Two 2-kg masses: $X=X_1+X_2$, $\text{E}(X)=1.980+1.980=3.960$
$\text{Var}(X)=0.015^2+0.015^2=0.00045$, $\text{sd}(X)=\sqrt{0.00045}$
One 4-kg mass $Y$: $\text{E}(Y)=3.940$, $\text{sd}(Y)=0.002$,
$\text{Var}(Y)=0.002^2=0.000004$
Let $W=X-Y$, $\text{E}(W)=3.960-3.940=0.020$
$\text{Var}(W)=0.00045+(-1)^2\times0.000004=0.000454$
$\text{sd}(W)=\sqrt{0.000454}$ .: $\Pr(W>0)\approx 0.826$ **C**

**SECTION B**

Q1a $f(x)=\dfrac{x^2}{x-1}=\dfrac{(x+1)(x-1)+1}{x-1}=x+1+\dfrac{1}{x-1}$
Asymptotes: $x=1$ and $y=x+1$

Q1b




Q1ci $f(x)=\frac{x^2}{x-k}=\frac{(x+k)(x-k)+k^2}{x-k}=x+k+\frac{k^2}{x-k}$

Asymptotes: $x=k$ and $y=x+k$

Q1cii $f'(x)=1-\left(\frac{k}{x-k}\right)^2=0$

.: $x=2k$ and $y=4k$ OR $x=0$ and $y=0$

Distance $=\sqrt{(2k)^2+(4k)^2}=\sqrt{20k^2}=2\sqrt{5}\,|k|$

Q1di Let $x+3=\left|\frac{x^2}{x-1}\right|$, $(x+3)^2=\left(\frac{x^2}{x-1}\right)^2$,

$(2x-3)(2x^2+2x-3)=0$, $x=\frac{3}{2}$, $x=\frac{-1\pm\sqrt{7}}{2}$

$$V=\int_{\frac{-\sqrt{7}-1}{2}}^{\frac{\sqrt{7}-1}{2}}\left(\pi(x+3)^2-\pi\left|\frac{x^2}{x-1}\right|^2\right)dx=\pi\int_{\frac{-\sqrt{7}-1}{2}}^{\frac{\sqrt{7}-1}{2}}\left((x+3)^2-\left(\frac{x^2}{x-1}\right)^2\right)dx$$

Q1dii By CAS $V=51.42$

Q2ai $u\,v=(a+i)(b-\sqrt{2}\,i)$

$=(ab+\sqrt{2})+(b-\sqrt{2}\,a)i=(\sqrt{2}+\sqrt{6})+(\sqrt{2}-\sqrt{6})i$

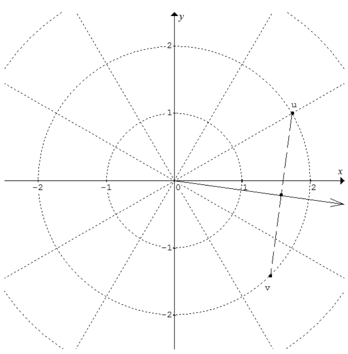
.: $ab=\sqrt{6}$ and $b-\sqrt{2}\,a=\sqrt{2}-\sqrt{6}$

Eliminate $b$: $a^2+(1-\sqrt{3})a-\sqrt{3}=0$

Q2aii $a^2+(1-\sqrt{3})a-\sqrt{3}=(a-\sqrt{3})(a+1)=0$

.: $a=-1$ and $b=-\sqrt{6}$

Q2b $u=\sqrt{3}+i=2\operatorname{cis}\frac{\pi}{6}$, $v=\sqrt{2}-\sqrt{2}\,i=2\operatorname{cis}\left(-\frac{\pi}{4}\right)$




Q2c $\theta=\frac{\operatorname{Arg}(u)+\operatorname{Arg}(v)}{2}=\frac{1}{2}\left(\frac{\pi}{6}-\frac{\pi}{4}\right)=-\frac{\pi}{24}$

Q2d $\angle uOv=\frac{\pi}{6}+\frac{\pi}{4}=\frac{5\pi}{12}$

Minor segment area = minor sector area – triangular area

$=\frac{5\pi}{6}-\frac{1}{2}\times2\times2\times\sin\frac{5\pi}{12}\approx0.69$

Q3ai $\int\frac{1}{2}e^x\,dx=\int\frac{1}{1+4t^2}\,dt$

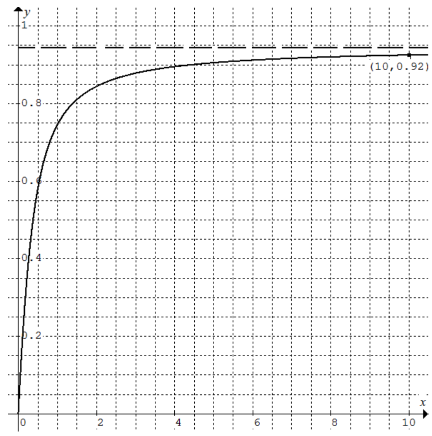
Q3aii $\frac{1}{2}\int e^x\,dx=\frac{1}{2}\int\frac{\frac{1}{2}}{\frac{1}{4}+t^2}\,dt$ .: $e^x=\tan^{-1}(2t)+c$

$x=0$, $t=0$ .: $e^x=\tan^{-1}(2t)+1$

Q3bi As $t\to\infty$, $\tan^{-1}(2t)\to\frac{\pi}{2}$, $x\to\log_e\left(\frac{\pi}{2}+1\right)$

.: asymptote is $x=\log_e\left(\frac{\pi}{2}+1\right)$

Q3bii $x(10)=\log_e(\tan^{-1}20+1)\approx0.92$




Q3c At $t=3$, $x=\log_e(\tan^{-1}6+1)$, $\frac{dx}{dt}=\frac{2e^{-x}}{1+4t^2}\approx0.02$ m s$^{-1}$

Q3d First particle: $x_1=\log_e(\tan^{-1}(2\times6)+1)=\log_e(\tan^{-1}12+1)$

Second particle: $x_2=\log_e(\tan^{-1}(3\times6-6)+1)=\log_e(\tan^{-1}12+1)=x_1$

Q3e Same distance at $t=6$. First particle: $\frac{dx}{dt}=\frac{\frac{2}{1+4t^2}}{\tan^{-1}2t+1}$

Second particle: $\frac{dx}{dt}=\frac{\frac{3}{1+(3t-6)^2}}{\tan^{-1}(3t-6)+1}$

Ratio is $\frac{v_1}{v_2}=\frac{2}{3}$ when the two particles are at the same distance from $O$.

Q4a $\dot{r} = \frac{\pi}{8}\cos\frac{\pi t}{4}\tilde{i} + 2\tilde{j}$, $\dot{r}(0) = \frac{\pi}{8}\tilde{i} + 2\tilde{j}$, $\tan\theta = \frac{\frac{\pi}{8}}{2}$, $\theta \approx 11.1°$

Q4bi Speed $= |\dot{r}(0)| = \sqrt{\left(\frac{\pi}{8}\right)^2 + 2^2} \approx 2.04$ m s$^{-1}$

Q4bii $|\dot{r}| = \sqrt{\left(\frac{\pi}{8}\right)^2 \cos^2\frac{\pi t}{4} + 4}$

Min. speed $= 2$ m s$^{-1}$ when $\cos^2\frac{\pi t}{4} = 0$, $t = 2$

Q4c Let $\tilde{d} = \tilde{r} - 7\tilde{j} = \frac{1}{2}\sin\frac{\pi t}{4}\tilde{i} + (2t-7)\tilde{j}$

$|\tilde{d}| = \sqrt{\frac{1}{4}\sin^2\frac{\pi t}{4} + (2t-7)^2}$, by CAS $\min|\tilde{d}| = 0.188$ m

Q4d Distance $= \int_0^4 \sqrt{\left(\frac{\pi}{8}\cos\frac{\pi t}{4}\right)^2 + 2^2}\, dt \approx 8.077$ m

Q5a $a = \frac{R}{m} = \frac{10\sqrt{2}\cos 45° + 20\sqrt{3}\cos 30° - 5k}{5}$

$= \frac{10 + 30 - 5k}{5} = 8 - k$ m s$^{-2}$

Q5b Note: The question did not define $t = 0$.
$\Delta p = m\Delta v = 5 \times (2 - 0.5) = 7.5$ kg m s$^{-1}$

Q5c $v - u = at$, $2 - 0.5 = (8-k)5$, $k = 7.7$

Q5d $v^2 = u^2 + 2as$, $2^2 = 0.5^2 + 2\times(8 - 7.7)\times s$, $s = 6.25$
Distance from $O$ to $P$ is 6.25 m

Q5e $a = \frac{R}{m} = \frac{-5g\sin\theta - 38.5}{5} = -g\sin\theta - 7.7$
$s = 0.2$, $u = 1.95$, $v = 0$
$v^2 = u^2 + 2as$, $0 = 1.95^2 + 2\times(-9.8\sin\theta - 7.7)\times 0.2$, $\theta \approx 10.6°$

Q6a $H_0$ is $\mu = 15$, $H_1$ is $\mu < 15$

Q6b $\mu = 15$, $\sigma = 0.25$

Sample: $n = 64$, $\bar{x} = 14.94$, $\text{sd}(\bar{X}) = \frac{0.25}{\sqrt{64}} = 0.03125$

$p = \Pr(\bar{X} < 14.94 \mid \mu = 15) \approx 0.027$

Q6c $p < 0.05$ .: good evidence against $H_0$ at 5% level of significance for the one-tailed test .: the supplier's claim is not supported.

Q6d $\Pr(\bar{X} < a \mid \mu = 15) > 0.05$, $\Pr\left(Z < \frac{a - 15}{0.03125}\right) > 0.05$,

$a \approx 14.95$

Q6e $\mu = 406$, $\sigma = 5$, $\text{Var}(X) = 25$
Let $Y = X_1 - X_2$ be the difference
.: $\mu_Y = \mu_{X_1} - \mu_{X_2} = 0$,
$\text{Var}(Y) = 1^2\,\text{Var}(X_1) + (-1)^2\,\text{Var}(X_2) = 25 + 25 = 50$, $\sigma_Y = \sqrt{50}$
$\Pr(-3 \le Y \le 3) \approx 0.329$

Q6f 375 mL $\equiv 375 \times 1.04 = 390$ grams
Let $X_1$ and $X_2$ be the masses of a can and soft drink respectively, and $Y = X_1 + X_2$ the mass of a can of soft drink.
Statistics for
$Y$: $\mu = 406$, $\text{Var}(Y) = 5^2 = 25$
$X_1$: $\mu = 15$, $\text{Var}(X_1) = 0.25^2 = 0.0625$
.: for $X_2 = Y - X_1$, $\mu = 406 - 15 = 391$.
Since $\text{Var}(Y) >>> \text{Var}(X_1)$
.: $\text{Var}(X_2) \approx \text{Var}(Y) = 25$,
.: $\Pr(X_2 < 390) \approx 0.421$

*Please inform mathline@itute.com re conceptual and/or mathematical errors*