itute.com

*Online & home tutors* *Registered business name: itute ABN: 96 297 924 083*

# 2023 Mathematical Methods

# Year 12

## Modelling Task

**Time allowed: 2 hours plus**

© itute 2021

# Modelling Task

**Theme:** Filling water tanks

**Assumed knowledge:** Functions, graphs, exponential and logarithmic functions, transformations, equations, calculus, rate of change, definite integral and CAS

**Information:** If $y = f(x)$ where $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.
The volume of 1 litre is 1000 cm$^3$.

**Part I (60 minutes plus)**

Water runs into a tank at 6 litres per minute.
At time $t$ seconds, let $V$ cm$^3$ be the volume and $h$ cm the depth of water in a tank.

Consider a cylindrical tank with a circular base of 50 cm radius and a height of 200 cm.
The tank is empty initially.

a. Show that $V = 100t$.

b. Show that $h = \frac{0.04}{\pi} t$.

c. Find the exact time (hours) required to fill the tank.

d. Verify that $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$.

e. If the time to fill the tank is 2 hours, find the exact rate (litres per minute) that water runs into the tank.


© itute 2021

Consider an inverted cone-shape tank. The height of the tank is 200 cm.
At time $t$ seconds, the depth of water in the tank is $h$ cm, the radius of the circular water surface is $r$ cm, and the volume of water in the tank is given by $V = \frac{1}{3}\pi r^2 h$ cm$^3$.

The relationship between $h$ and $r$ is $h = \frac{4\sqrt{3}}{3}r$.
Water runs into the tank at 6 litres per minute.

f. Find the exact time (hours) required to fill the tank.

g. Write an expression for the volume of water $V$ in the tank in terms of $h$ at time $t$.

h. Use $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ in part d to show that $\frac{dh}{dt} = \frac{1600}{3\pi h^2}$ for $h \in (0, 200)$.

i. Find the exact additional time (hours) required to completely fill the tank from a depth of 100 cm.

j. Find the exact rate of change in the radius (cm min$^{-1}$) of the circular water surface when the tank is filled to a depth of 100 cm.


© itute 2021

Water level rises when water runs into a tank.

In part b, $h = \frac{0.04}{\pi}t$ when water runs into the cylindrical tank at 6 litres per minute.

In part h, water level in the tank rises at a rate of $\frac{dh}{dt} = \frac{1600}{3\pi h^2}$ when water runs into the inverted cone-shape tank at 6 litres per minute.

k. Sketch the graph of $\frac{dh}{dt}$ versus $h$ in each case, for $h \in (10, 200)$

Compare and comment on the rates of change in the depth of water.

l. Now the water level is required to rise at a constant rate of $\alpha$ cm per minute.

Find, in terms of $h$ and $\alpha$, the rate of water running into the tank in litres per minute.

m. Find the exact value of $\alpha$ such that water runs into the tank at 10 litres per minute when the depth is 100 cm.

**End of Part I**


© itute 2021

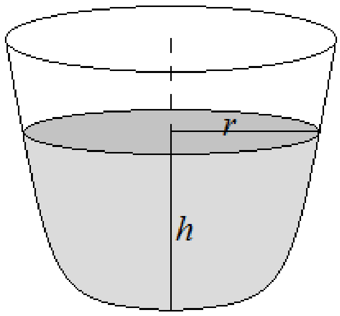
**Part II (60 minutes plus)**

In Part I, you modelled the filling of a cylindrical tank and an inverted cone-shape tank.

**Information:** If $y = f(x)$ where $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

The volume of 1 litre is 1000 $cm^3$.

Consider a tank of the shape shown below. Initially, the tank is empty.
At time $t$ seconds, let $V$ $cm^3$ be the volume and $h$ cm the depth of water in the tank.
The radius of the circular water surface is $r$ cm when the depth is $h$ cm.
Water runs into the tank at a constant rate of 6 litres per minute.




The relationship between $h$ and $r$ is $h = e^{kr} - 1$ where $k$ is a constant and $h \in [0, 200]$.

a. Select a value of $k \in (0.08, 0.12)$.
Sketch the graph of $h$ versus $r$ showing the relationship for your selected $k$ value.
Show the coordinates of the endpoints.

b. State the effect of increasing the value of $k$ on the graph of $h$ versus $r$, and hence the volume of the tank.

c. Express $r$ in terms of $h$ and $k$ in the relationship $h = e^{kr} - 1$.

 © itute 2021 

At time $t$ seconds, let $V = \frac{\pi}{k^2}\int_0^h (\log_e(h+1))^2 \, dh$ be the volume (cm$^3$) of water in the tank shown above.

d. Given that the tank holds $500000\pi$ cm$^3$ of water when full, show that $k \approx 0.08856$. Hence determine the radius of the top of the tank.

e. Find the time (hours) required to fill the tank to a depth of 100 cm, and the additional time required to completely fill the tank when water runs into the tank at 6 litres per minute.

f. Show that $\frac{dV}{dh} \approx \frac{500000\pi}{3921.2}(\log_e(h+1))^2$, and when water runs into the tank at 6 litres per minute, $\frac{dh}{dt} \approx \frac{3921.2}{5000\pi(\log_e(h+1))^2}$.

The result in part f shows that $\frac{dh}{dt}$ decreases as depth $h$ increases.

g. Let $\beta = \frac{dh}{dt}$ when $h = 9$. Calculate (i) $\beta$, and (ii) $h$ when $\frac{dh}{dt} = \frac{1}{4}\beta$.


© itute 2021

h. Select a suitable value of $h > 9$ and calculate (i) $\frac{dh}{dt}$ at your selected $h$ value, and (ii) $h$ when $\frac{dh}{dt} = \frac{1}{4} \times (\text{answer in part (i)})$.

i. Determine the exact maximum value of $h$ that you can select in part h.

Now water runs into the tank at a variable rate $\frac{dV}{dt}$ cm$^3$ per second such that the depth of water increases at a **constant** rate and the time required to fill the tank is $5000\pi$ seconds.

j. Show that $\frac{dV}{dt} \approx 5.1(\log_e(h+1))^2$.

k. Find $h$ cm when variable rate $\frac{dV}{dt} = \frac{1}{2} \times \left(\frac{dV}{dt}\right)_{\text{max}}$.
Correct your answer to 1 decimal place.

**End of Part II**


© itute 2021