# 2023 NSW ESA Mathematics Advanced Solutions



**Section I**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| D | D | A | B | A | C | A | B | D | C |

Q1 D

Q2 $(S, D)$:
$(4,3), (4,4), (4,5), (4,6), (3,4), (3,5), (3,6), (2,5), (2,6), (1,6)$

$\Pr = \frac{10}{24} = \frac{5}{12}$ D

Q3 $1 - x > 0$, $x < 1$ A

Q4 B

Q5 $\int_0^1 f(x)dx + \int_1^a f(x)dx = \int_0^a f(x)dx = 0$ $\therefore$ $-1 + \int_1^a f(x)dx = 0$

$\therefore$ $\int_1^a f(x)dx = 1$ $\therefore$ $\int_{-a}^1 f(x)dx = -1 + 1 - 1 = -1$ A

Q6 Consider $f'(x)$. C

Q7 $y'(5) = f'(g(5))g'(5) = f'(1)g'(5) = -8$. A

Q8 $\log_a x^3 = b$, $3\log_a x = b$, $\log_a x = \frac{b}{3}$, $x = a^{\frac{b}{3}}$ B

Q9 $f(x)f(-x) = f(-x)f(x)$ D

Q10 $y = \frac{x^2}{a^2} = \left(\frac{x}{a}\right)^2$ i.e. horizontal dilation by a factor of $a$

$\therefore$ $\overline{ST} = a\overline{PQ} = aL$ C

**Section II**

Q11 $a = 3$, $d = 4$, $t_{15} = 3 + (15 - 1)4 = 59$

Q12a $E(X) = 0\times 0 + 1\times 0.3 + 2\times 0.5 + 3\times 0.1 + 4\times 0.1 = 2$

Q12b
$Var(X) = 0^2\times 0 + 1^2\times 0.3 + 2^2\times 0.5 + 3^2\times 0.1 + 4^2\times 0.1 - 2^2 = 0.8$
$sd(X) = \sqrt{0.8} \approx 0.9$

Q13 $t = 0$, $P = 4000$, $P(t) = \frac{3000}{2}e^{2t} + c = 1500e^{2t} + 2500$

Q14 $m = \frac{dy}{dx} = 6(2x+1)^2$ At $(0,1)$, $m = 6$, tangent is $y = 6x + 1$

Q15a $13.181x = 450000$, $x \approx 34140$ dollars

Q15b Interest rate per quarter $= \frac{6\%}{4} = 1.5\% = 0.015$,
10 years $= 40$ periods
Amount $= 8535\times 54.268 = 463177.38$ dollars

Q16 Arc $PQ = \frac{110}{360}\times 2\pi\times 2.1 \approx 4.03171$

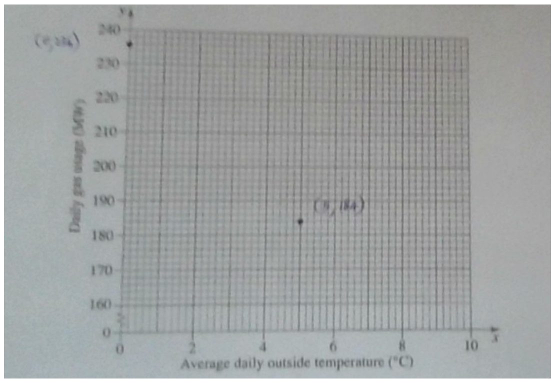
Line segment $PQ = 2\times 2.1\sin\frac{110°}{2} \approx 3.44044$

Perimeter $= 2(3.6 + 8.0) - \text{line}PQ + \text{arc}PQ \approx 23.8$ m

Q17 Let $u = x^2 + 1$, $\frac{du}{dx} = 2x$

$\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{3}u^{\frac{3}{2}} + c = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + c$

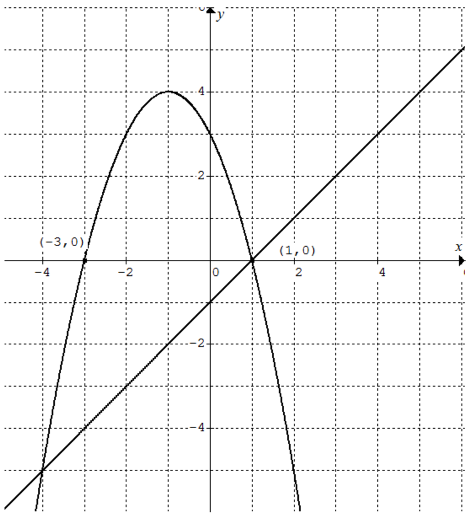
Q18a $\bar{x} = 5$, $\bar{y} = 184$ $(5, 184)$ $(0, 236)$




Q18b Gradient $= \frac{184 - 236}{5 - 0} = -10.4$, equation: $y = -10.4x + 236$

Q18c When $x = 23$, $y = -3.2$ i.e. negative gas usage, not making sense

Q19a




Q19b At the intersections, $x - 1 = (1 - x)(3 + x)$, $x = -4$ or $x = 1$
$x - 1 < (1 - x)(3 + x)$ for $-4 < x < 1$

Q20 $\theta - 60° = -60°, 240°, 300°$
$\therefore$ $\theta = 0°, 300°, 360°$

Q21 $t_4 = ar^3 = 48$, $t_8 = ar^7 = \frac{3}{16}$ $\therefore$ $\frac{t_8}{t_4} = r^4 = \frac{1}{16^2}$

$\therefore$ $r^2 = \frac{1}{16}$ $\therefore$ $r = \pm\frac{1}{4}$ and the corresponding $a = \pm 3072$

Q22 $\overline{AM} = \sqrt{7^2 + 3^2} = \sqrt{58}$, $\angle AEM = \tan^{-1}\left(\frac{\sqrt{58}}{8}\right) \approx 44°$

Q23 $z = \frac{11.93 - 10.40}{1.15} \approx 1.33$, $P(z < 1.33) = 0.9082$
Number expected to weigh less than 11.93 kg $= 400\times 0.9082 \approx 363$
$\therefore$ Number expected to weigh more than 11.93 kg $= 400 - 363 = 37$

Q24a $(y-1)(x-2)=50$ .: $y=\frac{50}{x-2}+1$

Q24b Area of concrete path

$=A=xy-50=x\left(\frac{50}{x-2}+1\right)-50=\frac{50x}{x-2}+x-50$ where $x>2$

$\frac{dA}{dx}=\frac{(x-2)50-50x}{(x-2)^2}+1=1-\frac{100}{(x-2)^2}$

Let $\frac{dA}{dx}=0$, simplify to $(x-2)^2=100$, $x=12$

Consider $\frac{dA}{dx}$ on both sides of $x=12$

When $x=11$, $\frac{dA}{dx}$ is negative; when $x=13$, $\frac{dA}{dx}$ is positive

.: $x=12$ gives a minimum area $A$

Q25a $A_1=10000(1.004)-M$

$A_2=(10000(1.004)-M)1.004-M=10000(1.004)^2-M(1.004)-M$

Q25b $A_3=10000(1.004)^3-M(1.004)^2-M(1.004)-M$

$A_n=10000(1.004)^n-M\left((1.004)^{n-1}+(1.004)^{n-2}+\cdots+1\right)$

$=10000(1.004)^n-M\left(\frac{(1.004)^n-1}{1.004-1}\right)=10000(1.004)^n-M\left(\frac{(1.004)^n-1}{0.004}\right)$

$=10000(1.004)^n-250M(1.004)^n+250M$

$=(10000-250M)(1.004)^n+250M$

Q25c Let $n=100$ and $A_n=0$

$(10000-250M)(1.004)^{100}+250M=0$,

$10000(1.004)^{100}-250M(1.004)^{100}+250M=0$

$10000(1.004)^{100}=250M\left((1.004)^{100}-1\right)$ .: $M=121.52$

Q26a $x==-1.5\pi\int\sin\frac{5\pi}{4}t\,dt=-1.5\pi\left(-\frac{4}{5\pi}\cos\frac{5\pi}{4}t\right)+c$

At $t=0$, $x=11.2$ .: $x=1.2\cos\frac{5\pi}{4}t+10$

Q26b Period $=\frac{2\pi}{\frac{5\pi}{4}}=\frac{8}{5}$, number of cycles in 10 s $=\frac{10}{\frac{8}{5}}=\frac{25}{4}=6\frac{1}{4}$

.: Number of times $=6$

Q27a Translations: $b=6$, $c=7$ .: $y=a|x-6|+7$

$(3,-5)$, .: $a=-4$

Q27b $m<\frac{7}{6}$ and $m>\frac{7-(-5)}{6-9}=-4$ .: $-4<m<\frac{7}{6}$

Q28 Let $\frac{dy}{dx}=3x^2-6x-8=1$ where $x>0$

.: $(x-3)(x+1)=0$ .: $x=3$

$y=x^3-3x^2-8x+c=x^3-3x^2-8x+2$ $\because$ $(-1,6)$ is on the curve

At $x=3$, $y=-22$ .: $R(3,-22)$

Q29a $f'(x)=24x-36x^2=0$, $x(2-3x)=0$ Mode $x=\frac{2}{3}$

Q29b $\int_0^x 12t^2(1-t)dt=4x^3-3x^4$, $0\le x\le 1$,

Cumulative distribution function $C(x)=\begin{cases}0 & x<0\\ 4x^3-3x^4 & 0\le x\le 1\\ 1 & x>1\end{cases}$

Q29c At the mode, $C\left(\frac{2}{3}\right)=\frac{16}{27}>0.5$ .: the mode is greater than the median.

Q30a $f'(x)=e^{-x}(\cos x-\sin x)=0$

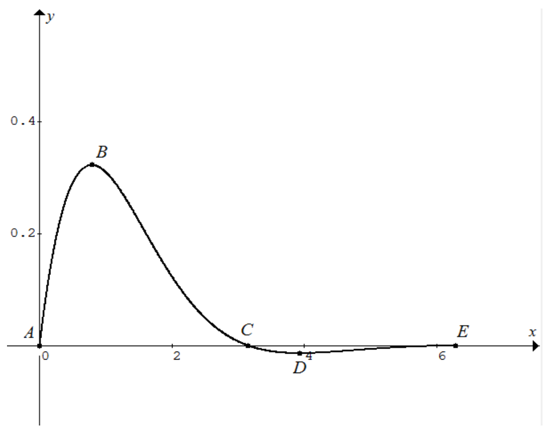
$e^{-x}>0$ .: $\cos x-\sin x=0$, $\tan x=1$ and $0\le x\le 2\pi$

.: $x=\frac{\pi}{4},\frac{5\pi}{4}$, correspondingly, $y=\frac{e^{-\frac{\pi}{4}}}{\sqrt{2}},-\frac{e^{-\frac{5\pi}{4}}}{\sqrt{2}}$

Stationary points $\left(\frac{\pi}{4},\frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4},-\frac{e^{-\frac{5\pi}{4}}}{\sqrt{2}}\right)$

Q30b $x$-intercepts: $f(x)=0$, $x=0,\pi,2\pi$

$A(0,0)$, $B\left(\frac{\pi}{4},\frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}\right)$, $C(\pi,0)$, $D\left(\frac{5\pi}{4},-\frac{e^{-\frac{5\pi}{4}}}{\sqrt{2}}\right)$, $E(2\pi,0)$




Q31a No $\because$ $P(F\mid S)\ne P(F)$

Q31b $P(S\mid F)=\frac{P(S\cap F)}{P(F)}$ .: $P(S\cap F)=\frac{1}{3}\times\frac{3}{10}=\frac{1}{10}$

Also $P(S\cap F)=P(F\mid S)P(S)$ .: $\frac{1}{10}=\frac{1}{8}P(S)$ .: $P(S)=\frac{4}{5}$

Q31c $P(S')=1-P(S)=\frac{1}{5}$

$P(\text{at least one } S')=1-P(\text{none } S')=1-(P(S))^4=1-\left(\frac{4}{5}\right)^4=\frac{369}{625}$

Q32a Area $=\int_0^{\ln 2}\left(e^{-2x}-\left(e^{-x}-\frac{1}{4}\right)\right)dx$

$=\left[-\frac{1}{2}e^{-2x}+e^{-x}+\frac{1}{4}x\right]_0^{\ln 2}=-\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\ln 2-\left(-\frac{1}{2}+1\right)$

$=\frac{1}{4}\ln 2-\frac{1}{8}$

Q32b At the intersections, $e^{-2x}=e^{-x}+k$ .: $\left(e^{-x}\right)^2-e^{-x}-k=0$

$e^{-x}=\frac{1\pm\sqrt{1+4k}}{2}$. Since $e^{-x}>0$, $\sqrt{1+4k}<1$ .: $k<0$

For two intersecting points (two values of $e^{-x}$), $\Delta=1+4k>0$

.: $k>-\frac{1}{4}$ Hence $-\frac{1}{4}<k<0$

*Please inform mathline@itute.com re conceptual and/or mathematical errors*