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2023 NSW ESA Mathematics Extension 1 Solutions

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Section I

1	2	3	4	5	6	7	8	9	10
D	A	C	C	B	C	B	A	D	B

Q1 $T(0)=15+4e^0=19$ D

Q2 $P\left(\hat{p}\geq\frac{9}{12}\right)=P\left(\hat{p}\geq\frac{3}{4}\right)$ A

Q3 C

Q4 Area $=10+2-3=9$ C

Q5 $\sin^{-1}(\sin a)=\sin^{-1}(\sin(\pi-a))=\pi-a$ B

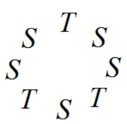
Q6 Projection of $10\tilde{a}$ onto $2\vec{b}$ = projection of $10\tilde{a}$ onto $\vec{b}$
$=10\times$ projection of $\tilde{a}$ onto $\vec{b}=10\tilde{c}$ C

Q7 $\sin^{-1}x$ is a strictly increasing function. B

Q8 The function is even .: it contains $|x|$. At $x=1$, $y=0$ A

Q9 $y=1$ cuts $y=|f(x)|$ at 4 points. D

Q10 The following diagram shows the only seating arrangement of students and teachers satisfying the requirement:



Number of ways $=5!\times 3!$ B

Section II

Q11a Eliminating t, $y=\frac{4}{3}x-\frac{4}{3}$

Q11b $\frac{10!}{3!2!}=302400$

Q11c $P(-1)=a-b-13=0$; $P(2)=4a+2b-4=-18$
$a=2$ and $b=-11$

Q11d $\int\frac{1}{\sqrt{4-9x^2}}dx=\int\frac{1}{\sqrt{9\left(\left(\frac{2}{3}\right)^2-x^2\right)}}dx=\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+c$

Q11e $\cos\theta+\sin\theta=1$ and $0\leq\theta\leq 2\pi$,
$2\cos^2\frac{\theta}{2}-1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=1$.: $\cos^2\frac{\theta}{2}-1+\sin\frac{\theta}{2}\cos\frac{\theta}{2}=0$,
$\sin\frac{\theta}{2}\cos\frac{\theta}{2}-\sin^2\frac{\theta}{2}=0$, $\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)=0$
.: $\sin\frac{\theta}{2}=0$ or $\sin\frac{\theta}{2}=\cos\frac{\theta}{2}$ where $0\leq\theta\leq 2\pi$.: $\theta=0,\ \frac{\pi}{2},\ 2\pi$

Q11f $n=900$, $p=0.3$.: $\mathrm{E}(\hat{p})=p=0.3$ and
$\mathrm{Var}(\hat{p})=\frac{0.3\times 0.7}{900}=\frac{7}{30000}$.: $\mathrm{sd}(\hat{p})=\sqrt{\frac{7}{30000}}\approx 0.0153$
$P(\hat{p}\leq 0.31)\approx P\left(z\leq\frac{0.31-0.3}{0.0153}\right)\approx P(z\leq 0.65)\approx 0.74$ from Table

Q12a Using $u=x-3$,
$\int_3^4(x+2)\sqrt{x-3}\,dx=\int_0^1(u+5)\sqrt{u}\,du=\int_0\left(u^{\frac{3}{2}}+5u^{\frac{1}{2}}\right)du$
$=\left[\frac{2u^{\frac{5}{2}}}{5}+\frac{10u^{\frac{3}{2}}}{3}\right]_0^1=\frac{56}{15}$

Q12b $n=1$, $2=2$, the statement is true
Assume the statement is true for $n=k$, i.e.
$(1\times 2)+(2\times 2^2)+(3\times 2^3)+\cdots+(k\times 2^k)=2+(k-1)2^{k+1}$
Now consider $n=k+1$,
$(1\times 2)+(2\times 2^2)+(3\times 2^3)+\cdots+(k\times 2^k)+((k+1)\times 2^{k+1})$
$=2+(k-1)2^{k+1}+(k+1)\times 2^{k+1}=2+2k\times 2^{k+1}=2+((k+1)-1)2^{(k+1)+1}$
The statement is also true .
.: The statement is true for integers $n\geq 1$

Q12ci X has a binomial distribution: $P(X=3)={}^5C_3 0.65^3 0.35^2$

Q12cii Let Y be the random variable number of rowing machine and it also has a binomial distribution, X and Y are independent.
$P(X=3\cap Y=0)=P(X=3)\times P(Y=0)={}^5C_3 0.65^3 0.35^2\times{}^4C_0 0.60^4$

Q12d ${}^{2022}C_{80}+{}^{2022}C_{81}+{}^{2023}C_{1943}$
$={}^{2022}C_{80}+{}^{2022}C_{81}+{}^{2023}C_{(2023-194)3}={}^{2022}C_{80}+{}^{2022}C_{81}+{}^{2023}C_{80}$
$={}^{2023}C_{81}+{}^{2023}C_{80}$ (by applying the given identity)
$={}^{2024}C_{81}$ (by applying the identity again)
.: $p=2024$ and $q=81$
Another possibility: $p=2024$ and $q=2024-81=1943$

Q12e $y=\frac{60}{x+5}$, $x=\frac{60}{y}-5=5\left(\frac{12}{y}-1\right)$
$V=\int_4^{12}\pi x^2dy+\pi\times 10^2\times 4=25\pi\int_4^{122}\left(\frac{144}{y^2}-\frac{24}{y}+1\right)dy+400\pi$
$=25\pi\left[-\frac{144}{y}-24\ln y+y\right]_4^{12}+400\pi=(1200-600\ln 3)\pi$

Q13ai Given $V=\pi\left(Rh^2-\frac{h^3}{3}\right)$ and $\frac{dV}{dt}=k(2R-h)$
$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$.: $\frac{dh}{dt}=\frac{\frac{dV}{dt}}{\frac{dV}{dh}}=\frac{k(2R-h)}{(2R-h)\pi h}=\frac{k}{\pi h}$

Q13aii $\frac{dh}{dt}=\frac{k}{\pi h}$, $\int dt=\int\frac{\pi h}{k}dh$.: $t=\frac{\pi h^2}{2k}$ given $h=0$ at $t=0$
When the tank is full, $h=R$, $t=\frac{\pi R^2}{2k}$ seconds

Q13aiii $\frac{dV}{dt}=-kh$ when the tank is full and $2kR=0$
.: $\frac{dh}{dt}=\frac{\frac{dV}{dt}}{\frac{dV}{dh}}=\frac{-k}{(2R-h)\pi}$
Empty when $t=-\frac{\pi}{k}\int_R^0(2R-h)dh=\frac{\pi}{k}\left[\frac{(2R-h)^2}{2}\right]_R^0=\frac{3\pi R^2}{2k}=3T$

Q13bi $\tan\theta = 2$ where $0 < \theta < \frac{\pi}{2}$, $\tan^2\theta = 4$, $\sec^2\theta = 5$,

$\sec\theta = \sqrt{5}$, $\cos\theta = \frac{1}{\sqrt{5}}$. $\sin\theta = \sqrt{1-\cos^2\theta} = \frac{2}{\sqrt{5}}$

At collision, $vt\cos\theta = ut$, $v\frac{1}{\sqrt{5}} = u$, $v = \sqrt{5}\,u$

Q13bii Same y - coordinate at collision time T, .: $vT\sin\theta = H$

$\sqrt{5}\,uT\frac{2}{\sqrt{5}} = H$, $T = \frac{H}{2u}$

Q13biii At collision $t = T = \frac{H}{2u}$, $\dot{r}_A = \begin{pmatrix} v\cos\theta \\ v\sin\theta - gt \end{pmatrix} = \begin{pmatrix} u \\ 2u - gT \end{pmatrix}$,

$\dot{r}_B = \begin{pmatrix} u \\ -gT \end{pmatrix}$, $\dot{r}_A.\dot{r}_B = 0$, $u^2 - 2ugT + g^2T^2 = 0$, $(u - gT) = 0$

.: $T = \frac{u}{g}$.: $\frac{H}{2u} = \frac{u}{g}$.: $H = \frac{2u^2}{g}$

Q13biv At maximum height $y_{\max}$, $2u - gt = 0$, $t = \frac{2u}{g}$

$y_{\max} = vt\sin\theta - \frac{1}{2}gt^2 = (\sqrt{5}\,u)\left(\frac{2u}{g}\right)\frac{2}{\sqrt{5}} - \frac{1}{2}g\left(\frac{2u}{g}\right)^2 = \frac{2u^2}{g} = H$

Q14ai $f(x) = 2x + \ln x$ where $x > 0$, $f'(x) = 2 + \frac{1}{x} > 0$ for $x > 0$

.: $f(x) = 2x + \ln x$ is a strictly increasing function for $x > 0$, and its inverse is a function.

Q14aii $f(1) = 2$, $(1,2)$ is a point of $f(x)$ and $(2,1)$ is the corresponding point on $g(x)$, $g'(2) = \frac{1}{f'(1)} = \frac{1}{3}$

Q14bi $y = \frac{1}{x}$, $(x-c)^2 + y^2 = c^2$, $x \neq 0$

By eliminating y, $x^4 - 2cx^3 + 1 = 0$, x-coordinates of intersections satisfy the equation .: zeros of the polynomial $P(x) = x^4 - 2cx^3 + 1$.

Q14bii Intersect at only one point when $P(x) = x^4 - 2cx^3 + 1$ has one x-intercept, i.e. its turning point is on the x-axis.

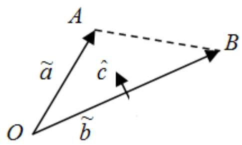
$P'(x) = 4x^3 - 6cx^2 = x^2(4x - 6c) = 0$, $x = \frac{3c}{2}$ and

$P\left(\frac{3c}{2}\right) = \left(\frac{3c}{2}\right)^4 - 2c\left(\frac{3c}{2}\right)^3 + 1 = 0$. Solve for $c = \frac{2}{27^{\frac{1}{4}}}$.

Q14ci $\overrightarrow{OB} = \tilde{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, let $\hat{c}$ be a unit vector perpendicular to $\tilde{b}$

.: $\hat{c} = \frac{\begin{pmatrix} b_2 \\ -b_1 \end{pmatrix}}{\sqrt{b_1^2 + b_2^2}}$ and the magnitude of the projection of $\overrightarrow{OA} = \tilde{a}$

onto $\hat{c}$ is given by $|\tilde{a}.\hat{c}| = \frac{\left|\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\begin{pmatrix} b_2 \\ -b_1 \end{pmatrix}\right|}{\sqrt{b_1^2 + b_2^2}} = \frac{|a_1b_2 - a_2b_1|}{\sqrt{b_1^2 + b_2^2}}$



Area of $\Delta OAB = \frac{1}{2} \times |\tilde{a}.\hat{c}| \times |\tilde{b}| = \frac{1}{2}|a_1b_2 - a_2b_1|$

Q14cii $\overrightarrow{OP} = \begin{pmatrix} r + r\cos t \\ r\sin t \end{pmatrix} = \begin{pmatrix} r(1+\cos t) \\ r\sin t \end{pmatrix}$,

$\overrightarrow{OQ} = \begin{pmatrix} -R + R\cos 2t \\ R\sin 2t \end{pmatrix} = \begin{pmatrix} R(-1+\cos 2t) \\ R\sin 2t \end{pmatrix}$ where $0 < t < \pi$

Let A = area of ΔOPQ

$= \frac{1}{2}|(r(1+\cos t)(R\sin 2t) - (r\sin t)R(-1+\cos 2t))|$

$= \frac{1}{2}rR|(1+\cos t)\sin 2t + \sin t(1-\cos 2t)|$

$= \frac{1}{2}rR|\sin 2t + \sin t + \sin 2t\cos t - \cos 2t\sin t|$

$= \frac{1}{2}rR|\sin 2t + \sin t + \sin(2t - t)|$

$= \frac{1}{2}rR|\sin 2t + 2\sin t| = rR|\sin t(\cos t + 1)| = rR\sin t(\cos t + 1)$

To max area, let $\frac{dA}{dt} = 0$, $\cos t(\cos t + 1) - \sin^2 t = 0$

.: $2\cos^2 t + \cos t - 1 = 0$, $(2\cos t - 1)(\cos t + 1) = 0$

Since $\cos t + 1 > 0$ for $0 < t < \pi$.: $2\cos t - 1 = 0$, $t = \frac{\pi}{3}$

Due to symmetry about the x-axis, $t = -\frac{\pi}{3}$ also.

Please inform mathline@itute.com re conceptual and/or mathematical errors.

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