2023 NSW ESA Mathematics Extension 2 Solutions

Section I

1	2	3	4	5	6	7	8	9	10
C	D	B	C	A	B	A	D	D	B

Q1 C

Q2 D

Q3 B

Q4 C

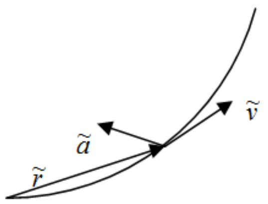
Q5 Consider choice A: equating the components, $3+\mu=-1-\lambda$, $1-\mu=5+\lambda$ and $-10-3\mu=2+3\lambda$ are consistent A

Q6 In each of A, C and D, both circular functions have the same period .: they describe SHMs B

Q7 A

Q8 Interpret $|z-1|$ and $|z-i|$ as distances of z from 1 and i respectively, any z in the shaded region satisfy choice D. D

Q9 See diagram below: $\tilde{r}, \tilde{v}, \tilde{a}$ satisfy the description of the motion. D



Q10 Choice B: B

All three vectors have the same length r, radius of the sphere.

The three conditions are $r^2\cos\alpha=1$, $r^2\cos\beta=2$ and $r^2\cos\gamma=3$

.: $\dfrac{1}{\cos\alpha}=\dfrac{2}{\cos\beta}=\dfrac{3}{\cos\gamma}$

There exist α,β,γ satisfying the conditions.

For example when $\cos\alpha=0.1$, $\cos\beta=0.2$ and $\cos\gamma=0.3$, and $r=\sqrt{10}$

Section II

Q11a $z=\dfrac{3\pm\sqrt{-7}}{2}=\dfrac{3}{2}\pm i\dfrac{\sqrt{7}}{2}$

Q11b $\tilde{a}.\tilde{b}=ab\cos\theta$, $1=7\sqrt{6}\cos\theta$, $\theta\approx 87°$

Q11c $\overrightarrow{BA}=-3\tilde{i}-\tilde{j}+2\tilde{k}$

$\tilde{r}=2\tilde{j}+3\tilde{k}+\lambda\left(-3\tilde{i}-\tilde{j}+2\tilde{k}\right)$, $\tilde{r}=-3\lambda\tilde{i}+(2-\lambda)\tilde{j}+(3+2\lambda)\tilde{k}$

Q11d $ABCD$ and $ABEF$ are parallelograms.

.: $\overrightarrow{AB}\,//\,\overrightarrow{DC}$ and $AB=DC$, $\overrightarrow{AB}\,//\,\overrightarrow{FE}$ and $AB=FE$

.: $\overrightarrow{DC}\,//\,\overrightarrow{FE}$ and $DC=FE$.: $CDFE$ is a parallelogram

Q11e Period $=\dfrac{2\pi}{3}$ and centre $x=4$

Q11f $\displaystyle\int_0^2 \frac{2}{x+1}+\frac{3}{x-3}\,dx=\left[2\ln|x+1|+3\ln|x-3|\right]_0^2=-\ln 3$

Q12a Assume $\sqrt{23}=\dfrac{a}{b}$ where a,b are integers.

.: $\sqrt{23}\,b=a$ is an integer. But $\sqrt{16}<\sqrt{23}<\sqrt{25}$ i.e. $4<\sqrt{23}<5$

.: $\sqrt{23}$ is not an integer because there is no integer between 4 and 5

.: $\sqrt{23}\,b$ cannot be an integer. The assumption led to a contradiction.

.: $\sqrt{23}$ cannot be rational.

Q12b $0\le(x-y)^2$, $2xy\le x^2+y^2$, $\dfrac{2xy}{x^2+y^2}\le 1$, $1+\dfrac{2xy}{x^2+y^2}\le 2$

.: $\dfrac{(x+y)^2}{x^2+y^2}\le 2$

Q12ci $\tilde{F}=-(mg\sin\theta)\tilde{i}+(R-mg\cos\theta)\tilde{j}$

Since there is zero acceleration in the $\tilde{j}$ direction .: $R-mg\cos\theta=0$

.: $\tilde{F}=-(mg\sin\theta)\tilde{i}$

Q12cii $\tilde{a}=\dfrac{\tilde{F}}{m}=-g\sin\theta\,\tilde{i}$

$\tilde{v}=\int\tilde{a}\,dt=-gt\sin\theta\,\tilde{i}+\tilde{c}=-gt\sin\theta\,\tilde{i}$ since the object is initially at rest.

Q12d Let $z^3=2-2i=2^{\frac{3}{2}}e^{\left(2k\pi-\frac{\pi}{4}\right)i}$.: $z=2^{\frac{1}{2}}e^{\frac{1}{3}\left(2k\pi-\frac{\pi}{4}\right)}$

Let $k=-1,0,1$ to obtain $z_1=2^{\frac{1}{2}}e^{-\frac{3\pi}{4}i}, z_2=2^{\frac{1}{2}}e^{-\frac{\pi}{12}i}, z_3=2^{\frac{1}{2}}e^{\frac{7\pi}{12}i}$ respectively

Q12ei Real coefficient, complex conjugate $2-i$ is also a zero.

Q12eii $P(z)=z^4-3z^3+cz^2+dz-30$

Let α and β be the other two zeros.

$\alpha+\beta+(2-i)+(2+i)=3$ and $\alpha\beta(2-i)(2+i)=-30$

Solve to find the other zeros to be -3 and 2.

Q13a $\displaystyle\int\frac{1-x}{\sqrt{5-4x-x^2}}\,dx=\int\frac{3}{\sqrt{9-(2+x)^2}}\,dx+\frac{1}{2}\int\frac{-4-2x}{\sqrt{5-4x-x^2}}\,dx$

$=3\sin^{-1}\left(\dfrac{2+x}{3}\right)+\sqrt{5-4x-x^2}+c$

Q13bi Given $k\ge 3$.: $k-3\ge 0$ and $k+1>0$

.: $(k-3)(k+1)\ge 0$ i.e. $k^2-2k-3\ge 0$

Q13bii Prove $2^n\ge n^2-2$ for all integers $n\ge 3$ by induction.

For $n=3$, $8\ge 7$ is true

For $n=k$, assume $2^k\ge k^2-2$ to prove that $2^{k+1}\ge(k+1)^2-2$.

Proof: $2^{k+1}=2\times 2^k\ge 2\left(k^2-2\right)$ (see assumption)

$2\left(k^2-2\right)=k^2+k^2-4\ge k^2+(2k+3)-4$ (from part i)

$k^2+(2k+3)-4=(k+1)^2-2$

.: $2^{k+1}\ge(k+1)^2-2$ is true.

Hence $2^n\ge n^2-2$ for all integers $n\ge 3$.

Q13ci $\tilde{v}(0)=\begin{pmatrix}\dot{x}(0)\\ \dot{y}(0)\end{pmatrix}=\begin{pmatrix}40\cos 30°\\ 40\sin 30°\end{pmatrix}=\begin{pmatrix}20\sqrt{3}\\ 20\end{pmatrix}$

Q13cii $\tilde{a}=\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix}=\begin{pmatrix}-4\dot{x}\\ -10-4\dot{y}\end{pmatrix}$

$\therefore\ \frac{d\dot{x}}{dt}=-4\dot{x}$ and $\frac{d\dot{y}}{dt}=-10-4\dot{y}$; $\int\frac{d\dot{x}}{\dot{x}}=-4\int dt$, $\int\frac{d\dot{y}}{\frac{5}{2}+\dot{y}}=-4\int dt$,

$\dot{x}(0)=20\sqrt{3}$ and $\dot{y}(0)=20$

$\therefore\ \ln\frac{\dot{x}}{20\sqrt{3}}=-4t$ and $\ln\frac{\frac{5}{2}+\dot{y}}{\frac{45}{2}}=-4t$

Hence $\dot{x}=20\sqrt{3}\,e^{-4t}$, $\dot{y}=\frac{45}{2}e^{-4t}-\frac{5}{2}$ and $\tilde{v}(t)=\begin{pmatrix}20\sqrt{3}\,e^{-4t}\\ \frac{45}{2}e^{-4t}-\frac{5}{2}\end{pmatrix}$.

Q13ciii $\dot{x}=20\sqrt{3}\,e^{-4t}$ and $x(0)=0$; $\dot{y}=\frac{45}{2}e^{-4t}-\frac{5}{2}$ and $y(0)=0$

$\therefore\ x=20\sqrt{3}\int e^{-4t}dt=5\sqrt{3}\left(1-e^{-4t}\right)$,

$y=\int\left(\frac{45}{2}e^{-4t}-\frac{5}{2}\right)dt=\frac{45}{8}\left(1-e^{-4t}\right)-\frac{5}{2}t\ \therefore\ \tilde{r}(t)=\begin{pmatrix}5\sqrt{3}\left(1-e^{-4t}\right)\\ \frac{45}{8}\left(1-e^{-4t}\right)-\frac{5}{2}t\end{pmatrix}$

Q13civ Range, when $\frac{45}{8}\left(1-e^{-4t}\right)-\frac{5}{2}t=0$, i.e. $1-e^{-4t}=\frac{4t}{9}$

From the given graph, it occurs when $1-e^{-4t}\approx 1$ and

$x=5\sqrt{3}\left(1-e^{-4t}\right)\approx 5\sqrt{3}\approx 8.7$ ∴ the horizontal range ≈ 8.7 m

Q14ai $z=e^{i\frac{\pi}{6}}=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+i\frac{1}{2}$, $w=e^{i\frac{3\pi}{4}}=-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$

$z+w=\left(\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\right)+i\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)$

$|z+w|^2=\left(\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)^2=\frac{4-\sqrt{6}+\sqrt{2}}{2}$

Q14aii $OACB$ is a rhombus with $OA=OB=1$ and

$OC^2=\frac{4-\sqrt{6}+\sqrt{2}}{2}$.

$\angle AOC=\frac{1}{2}\angle AOB=\frac{1}{2}\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)=\frac{7\pi}{24}$

Q14aiii Cosine rule:

$\cos\frac{7\pi}{24}=\frac{OC^2+OA^2-AC^2}{2.OA.OC}=\frac{OC^2+OA^2-OB^2}{2.OA.OC}=\frac{OC}{2}$

$=\frac{1}{2}\sqrt{\frac{4-\sqrt{6}+\sqrt{2}}{2}}=\frac{\sqrt{8-2\sqrt{6}+2\sqrt{2}}}{4}$

Q14b Period 8π , $x_1=4\cos\frac{t}{4}$ and $x_2=4\cos\frac{t-2\pi}{4}=4\sin\frac{t}{4}$

where $t\geq 2\pi$

Collision: $x_1=x_2$, $\cos\frac{t}{4}=\sin\frac{t}{4}$, first occurs when $\frac{t}{4}=\frac{5\pi}{4}$, $t=5\pi$

at $x_1=4\cos\frac{5\pi}{4}=-2\sqrt{2}$

Q14ci $a=\frac{F}{m}=-\left(g+kv^2\right)$,

$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=-\left(g+kv^2\right)$, $\int dx=\frac{1}{2}\int\frac{d\left(v^2\right)}{-\left(g+kv^2\right)}$,

$x=-\frac{1}{2k}\ln\left(g+kv^2\right)+c$ and $x=0$, $v=v_0$

$\therefore\ c=\frac{1}{2k}\ln\left(g+kv_0{}^2\right)$ $\therefore\ x=\frac{1}{2k}\ln\left(\frac{g+kv_0{}^2}{g+kv^2}\right)$

At max height H , $v=0$, $H=\frac{1}{2k}\ln\left(\frac{kv_0{}^2+g}{g}\right)$

Q14bii $a=\frac{F}{m}=g-kv^2$,

$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=g-kv^2$, $\int dx=\frac{1}{2}\int\frac{d\left(v^2\right)}{g-kv^2}$,

$x=-\frac{1}{2k}\ln\left(g-kv^2\right)+c$ and $x=0$, $v=0$

$\therefore\ c=\frac{1}{2k}\ln(g)$ $\therefore\ x=\frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$

After falling $x=H=\frac{1}{2k}\ln\left(\frac{kv_0{}^2+g}{g}\right)$ to reach ground at $v=v_1$,

$\frac{1}{2k}\ln\left(\frac{kv_0{}^2+g}{g}\right)=\frac{1}{2k}\ln\left(\frac{g}{g-kv_1{}^2}\right)$, $\frac{kv_0{}^2+g}{g}=\frac{g}{g-kv_1{}^2}$

$\therefore\ g\left(v_0{}^2-v_1{}^2\right)=kv_0{}^2v_1{}^2$

Q15ai $J_n=\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta$, $n\geq 0$

$J_n=\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\sin\theta\,d\theta=\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\frac{d}{d\theta}(-\cos\theta)d\theta$

$=\left[-\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}(n-1)\cos^2\theta\sin^{n-2}\theta\,d\theta$

$=0+(n-1)\int_0^{\frac{\pi}{2}}\left(1-\sin^2\theta\right)\sin^{n-2}\theta\,d\theta$

$=(n-1)\int_0^{\frac{\pi}{2}}\left(\sin^{n-2}\theta-\sin^n\theta\right)d\theta$

$=(n-1)\left(J_{n-2}-J_n\right)$

$\therefore\ J_n=(n-1)J_{n-2}-nJ_n+J_n$ $\therefore\ J_n=\frac{n-1}{n}J_{n-2}$

Q15aii $I_n=\int_0^1 x^n(1-x)^n\,dx$, $n\geq 1$

Let $x=\sin^2\theta$, $\frac{dx}{d\theta}=2\sin\theta\cos\theta$

When $x=0$, $\theta=0$ and when $x=1$, $\theta=\frac{\pi}{2}$

$I_n=\int_0^{\frac{\pi}{2}}2\sin^{2n+1}\theta\cos^{2n+1}\theta\,d\theta=\frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}}2^{2n+1}\sin^{2n+1}\theta\cos^{2n+1}\theta\,d\theta$

$=\frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}}\sin^{2n+1}2\theta\,d\theta=\frac{1}{2^{2n}}\int_0^{\pi}\frac{1}{2}\sin^{2n+1}\phi\,d\phi$ where $\phi=2\theta$

Since $\sin^{2n+1}\phi$ is symmetric about $\phi = \frac{\pi}{2}$

$\therefore\ I_n = \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\phi \, d\phi = \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta$, replacing ϕ by θ

Q15aiii $I_n = \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta \, d\theta = \frac{1}{2^{2n}} J_{2n+1}$

Since $J_n = \frac{n-1}{n} J_{n-2}$, replace n by $2n+1$ $\therefore\ J_{2n+1} = \frac{2n}{2n+1} J_{2n-1}$

and $I_n = \frac{1}{2^{2n}} \cdot \frac{2n}{2n+1} J_{2n-1}$

Replace n by $n-1$ in $I_n == \frac{1}{2^{2n}} J_{2n+1}$

$I_{n-1} = \frac{1}{2^{2(n-1)}} J_{2n-1}$ $\therefore\ J_{2n-1} = 2^{2(n-1)} I_{n-1}$

$\therefore\ I_n = \frac{1}{2^{2n}} \cdot \frac{2n}{2n+1} \cdot 2^{2n-2} I_{n-1}$ $\therefore\ I_n = \frac{n}{4n+2} I_{n-1}$

Q15bi $\overrightarrow{LP} = \overrightarrow{AP} - \overrightarrow{AL} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{AC}\right) - \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\left(-\tilde{b} + \tilde{c} + \tilde{d}\right)$

Q15bii

$\text{LHS} = b^2 + c^2 + d^2 + \left|\tilde{c} - \tilde{b}\right|^2 + \left|\tilde{d} - \tilde{b}\right|^2 + \left|\tilde{d} - \tilde{c}\right|^2$

$= 3\tilde{b} + 3\tilde{c} + 3\tilde{d} - 2\tilde{b}.\tilde{c} - 2\tilde{d}.\tilde{b} - 2\tilde{d}.\tilde{c}$

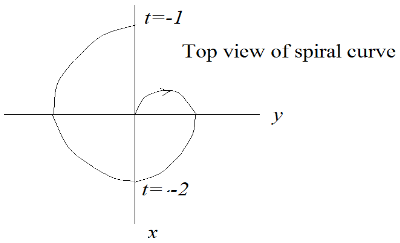
$\text{RHS} = \left|-\tilde{b} + \tilde{c} + \tilde{d}\right|^2 + \left|\tilde{b} - \tilde{c} + \tilde{d}\right|^2 + \left|\tilde{b} + \tilde{c} - \tilde{d}\right|^2$

$= \left(-\tilde{b} + \tilde{c} + \tilde{d}\right).\left(-\tilde{b} + \tilde{c} + \tilde{d}\right) + \left(\tilde{b} - \tilde{c} + \tilde{d}\right).\left(\tilde{b} - \tilde{c} + \tilde{d}\right) + \left(\tilde{b} + \tilde{c} - \tilde{d}\right).\left(\tilde{b} + \tilde{c} - \tilde{d}\right)$

Expand and simplify

$\text{RHS} = 3\tilde{b} + 3\tilde{c} + 3\tilde{d} - 2\tilde{b}.\tilde{c} - 2\tilde{d}.\tilde{b} - 2\tilde{d}.\tilde{c} = \text{LHS}$

Q15c Period $= 2$, $-3 \le t \le 3$



At $t = -2.5$, $y > 0$; $t = -2$, $y = 0$; $t = -1.5$, $x = 0$; $t = -1$, $y = 0$

$x = \sqrt{9 - t^2}\cos \pi t$, $y = -\sqrt{9 - t^2}\sin \pi t$, $z = t$ for $-3 \le t \le 3$

Q16ai The roots of $z^3 - 1 = 0$ are 1, w and w^2.

Sum of roots $1 + w + w^2 = 0$.

Q16aii $w = e^{i\frac{2\pi}{3}}$ rotates a complex number by $\frac{2\pi}{3}$ anticlockwise

Consider any anticlockwise equilateral ΔABC with its centre at the origin.

$aw = b$, $aw^2 = c$, $bw = c$, $bw^2 = a$, $cw = a$, $cw^2 = b$, $w^2 = -1 - w$

$(a + b + c)(1 + w + w^2) = 0$,

$a + aw + aw^2 + b + bw + bw^2 + c + cw + cw^2 = 0$

$a + aw + a(-1 - w) + b + bw + b(-1 - w) + bw + a + cw^2 = 0$

$\therefore\ a + bw + cw^2 = 0$

For a general anticlockwise equilateral with its centre translated by d from O (every vertex is translated by d):

$a' + b'w + c'w^2$

$= (a - d) + (b - d)w + (c - d)w^2$

$= a + bw + cw^2 - d(1 + w + w^2)$

$= a + bw + cw^2 = 0$

$\therefore\ a + bw + cw^2 = 0$ is true for any anticlockwise equilateral ΔABC.

Q16aiii $w + w^2 = -1$, $w^3 = 1$, $w^4 = w$

Consider $(a + bw + cw^2)(a + bw^2 + cw) = 0$, expand

$a^2 + abw^2 + acw + abw + b^2w^3 + bcw^2 + acw^2 + bcw^4 + c^2w^3 = 0$

$a^2 + abw^2 + acw + abw + b^2 + bcw^2 + acw^2 + bcw + c^2 = 0$

$\therefore\ a^2 + b^2 + c^2 = -ab(w + w^2) - bc(w + w^2) - ca(w + w^2)$

$\therefore\ a^2 + b^2 + c^2 = ab + bc + ca$

Q16bi Consider $f(x) = e^x$ and $g(x) = x$, $f(0) = 1$, $g(0) = 0$, $f'(0) = 1$ same as $g'(x) = 1$

For $x > 0$, $f'(x) = e^x > 1$ $\therefore\ e^x > x$ $\therefore\ x > \ln x$

Q16bii Prove $e^{n^2+n} > (n!)^2$ by induction:

For $n = 1$, $e^2 > 2 > 1$ is true

Assume it is true for $n = k$, i.e. $e^{k^2+k} > (k!)^2$

For $n = k + 1$, $e^{(k+1)^2+k+1} = e^{k^2+k+2(k+1)} = e^{k^2+k}e^{2(k+1)} = e^{k^2+k}\left(e^{k+1}\right)^2$

$\therefore\ e^{(k+1)^2+k+1} > (k!)^2(k+1)^2 = ((k+1)!)^2$ is true

$\therefore\ e^{n^2+n} > (n!)^2$ is true for $n \ge 1$

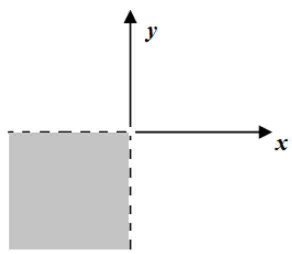
Q16c Given $|w| = |z| = 1$ and $\frac{\pi}{2} < \text{Arg}\left(\frac{z}{w}\right) < \pi$

$\therefore\ \frac{z}{w}$ lies on the unit circle in the second quadrant

$\frac{xz + yw}{z} = x + y\left(\frac{w}{z}\right) = x + y\overline{\left(\frac{z}{w}\right)}$ where $\overline{\left(\frac{z}{w}\right)}$ is the conjugate of $\frac{z}{w}$ lying in the third quadrant.

For $\frac{\pi}{2} < \text{Arg}\left(\frac{xz + yw}{z}\right) < \pi$, $\frac{xz + yw}{z} = x + y\overline{\left(\frac{z}{w}\right)}$ is required to be in the second quadrant also.

$\therefore\ y$ is a negative value to bring $\overline{\left(\frac{z}{w}\right)}$ to the first quadrant, and then add x (a negative value) to bring it to the second quadrant.



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