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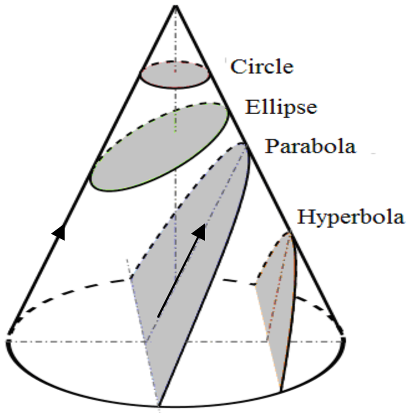
2023 Specialist Mathematics

Year 12 Application Task

(Time allowed: 4 hours plus)

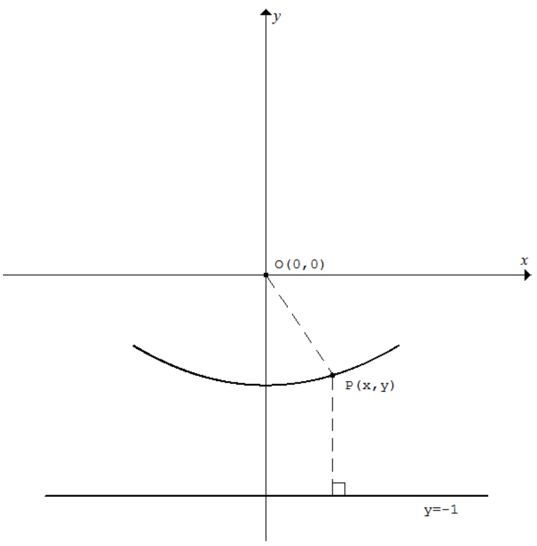
Theme: Investigation of some properties of CONIC SECTIONS



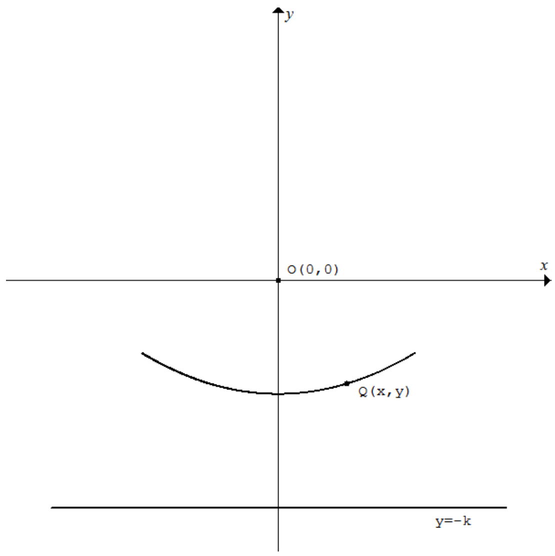
Assumed knowledge: Algebra; parabolas, ellipses and hyperbolas; transformations; similarity; circular functions; calculus; arc lengths and areas; CAS

Part I (85-95 min)

P (x, y) is a point on a curve as shown in Graph 1. The graph shows only a section of the curve.



Graph 1



Graph 2

a. Point P is equidistant (i.e. same distance) from the origin O $(0, 0)$ and from the line $y = -1$.

Use the above information to write an equation, hence show that the equation of the curve is $y = \frac{1}{2}x^2 - \frac{1}{2}$.

b. In Graph 2, point Q is on a curve and it is equidistant from the origin O $(0, 0)$ and from the line $y = -k$ where $k \in R^+$ and $k \neq 1$

Use the above information to write an equation, hence show that the equation of the curve is $y = \frac{1}{2k}x^2 - \frac{k}{2}$.

c. Plane figures (2-D shapes) are **similar** if a figure is the dilation of another figure in both x and y directions.

Use the above definition for similar figures, show that curve $y = \frac{1}{2k}x^2 - \frac{k}{2}$ is similar to curve $y = \frac{1}{2}x^2 - \frac{1}{2}$.

d. For this part only, consider point P at $x = 0.3$ on the curve $y = \frac{1}{2}x^2 - \frac{1}{2}$.

A vertical line passes through point P.
Find the gradient of the normal line to the curve at point P.
Hence find the acute angle between the normal line and the vertical line, and the acute angle between the normal line and line OP, in radians and correct to 2 decimal places. (Note: $\tan\theta =$ gradient)

e. Now consider any point P (x, y) on curve $y = \frac{1}{2}x^2 - \frac{1}{2}$.

A vertical line passes through point P.

(i) Show that the acute angle between the normal line at point P and the vertical line is $\left(\frac{\pi}{2} - \tan^{-1}\left(-\frac{1}{x}\right)\right)$.

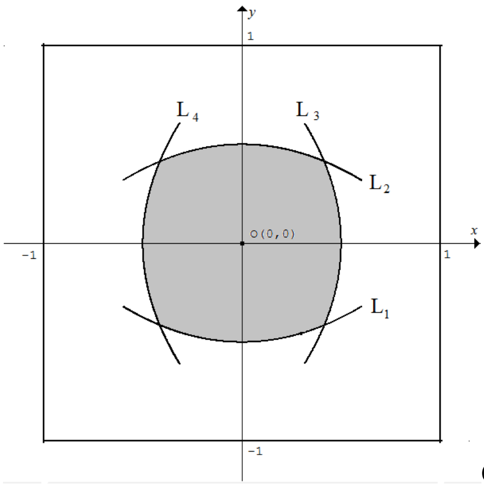
(ii) Show that the acute angle between the normal line and line OP is $\tan^{-1} x$.

(iii) Show that the two acute angles are equal.

When a light ray hits a surface at angle θ with the normal (to the surface), it reflects at the same angle on the other side of the normal.

(iv) Consider the curve as a reflecting surface and light rays are emitted from the origin $O(0, 0)$ in all directions. Write a brief statement about the reflected light rays from the curve.

In Graph 3 below the shaded region enclosed by curves L_1, L_2, L_3 and L_4 is a **regular** shape.



Graph 3

The equation of curve L_1 is $y = \frac{1}{2}x^2 - \frac{1}{2}$ as in part a.

f. Use suitable methods to find the equations of L_2, L_3 and L_4.

g. Use a suitable method to find the exact coordinates of the intersection of L_1 and L_3.
Hence write down the exact coordinates of the other three intersections.

h. Find the perimeter (correct to 1 decimal place) of the shaded region in Graph 3.

i. Find the area (correct to 1 decimal place) of the shaded region in Graph 3.

j. If the shaded region is enclosed by curves L_1, L_2, L_3 and L_4, and the equation of L_1 is $y=\frac{1}{2k}x^2-\frac{k}{2}$ as in part b, use a suitable method to determine the perimeter and area of the shaded region in terms of $k \in R^+$.

End of Part I

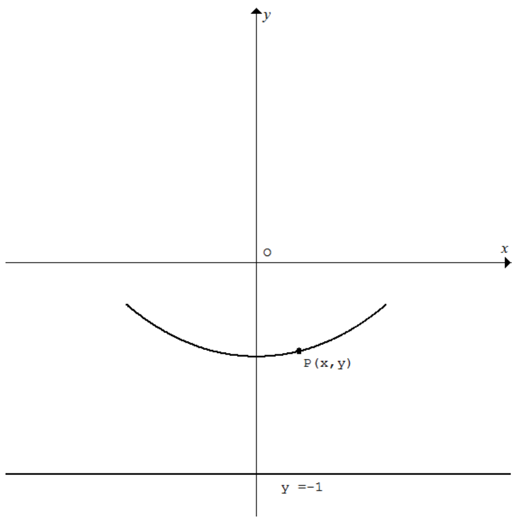
Part II (85-95 min)

P $(x,\ y)$ is a point on a curve as shown in Graph 4.

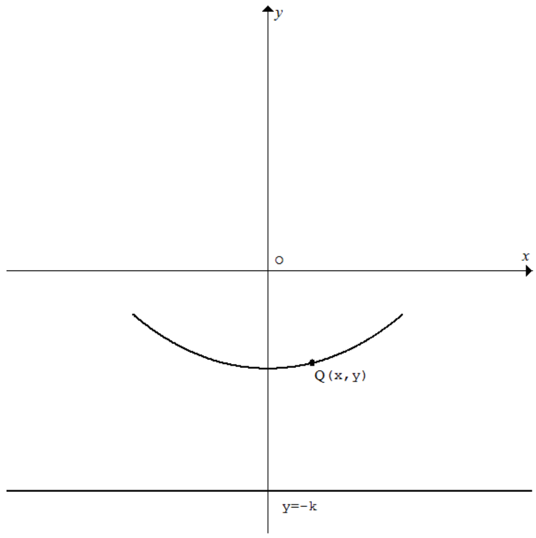
The distance of point P from the origin O $(0,0)$ is n times its distance from the line $y=-1$, where $0<n<1$.

Note: $n=1$ in **Part I**.

The Graph 4 below shows only a section of the curve.



Graph 4



Graph 5

a. Let $n=\frac{4}{5}$, i.e. the distance of point P from the origin O $(0,0)$ is $\frac{4}{5}$ of its distance from the line $y=-1$.

Use the above information to write an equation, hence show that the equation of the curve can be written as

$$\frac{x^2}{\left(\frac{3}{4}\right)^2}+\frac{\left(y-\frac{16}{9}\right)^2}{\left(\frac{20}{9}\right)^2}=1.$$

b. Name the curve in part a. Find the exact y-intercepts and coordinates of the centre of the curve.

c. Exactly the same curve $\frac{x^2}{\left(\frac{3}{4}\right)^2}+\frac{\left(y-\frac{16}{9}\right)^2}{\left(\frac{20}{9}\right)^2}=1$ can also be obtained by repeating part a with the following changes: $\mathrm{O}(0,0)$ is changed to $\mathrm{F}(a,b)$, line $y=-1$ is changed to $y=c$.

Explain why $a=0$, $b=\frac{32}{9}$ and $c=\frac{41}{9}$.

d. Let the coordinates of P be $\left(-\frac{4}{5},0\right)$. $\overline{\mathrm{OP}}$ is the distance of P from O and $\overline{\mathrm{FP}}$ is the distance of P from F.

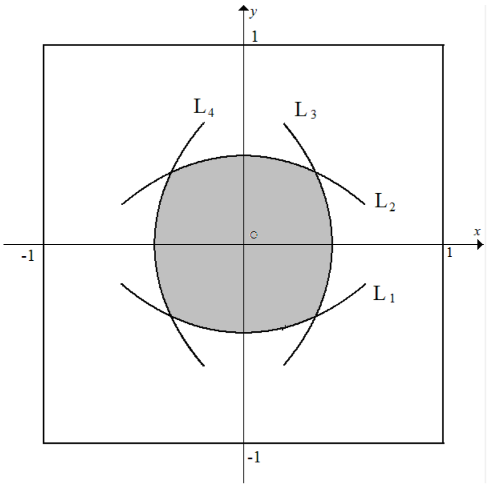
(i) Calculate the exact value of $\overline{\mathrm{OP}}+\overline{\mathrm{FP}}$.

(ii) $\mathrm{P}(x, y)$ is any point on the curve. Find the exact value of $\overline{\mathrm{OP}}+\overline{\mathrm{FP}}$.

Hint: Change $\overline{\mathrm{OP}}+\overline{\mathrm{FP}}$ to a function of a single variable and graph it over a suitable domain.

(iii) Write a brief statement to summarise your findings in parts d (i) and d (ii).

In Graph 6 below the shaded region enclosed by curves L_1, L_2, L_3 and L_4 is a regular shape.



Graph 6

The equation of curve L_1 is $\frac{x^2}{\left(\frac{3}{4}\right)^2}+\frac{\left(y-\frac{16}{9}\right)^2}{\left(\frac{20}{9}\right)^2}=1$ as in part a.

e. Find the equations of L_2, L_3 and L_4.

f. Find the exact coordinates of the intersection of L_1 and L_3.

g. Find the area (correct to 1 decimal place) of the shaded region in Graph 6.

Graph 6 is obtained by choosing $n=\frac{4}{5}$. L_1 and L_3 do not join smoothly at the intersection, i.e. they do not have the same gradient at the intersection. The same is true for the other intersections.

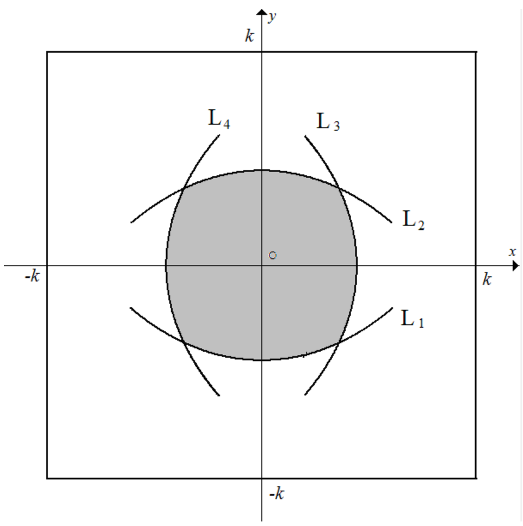
h. Investigate whether a value of n exists such that each intersection is a **smooth** joint. Otherwise explain why or prove that it does not exist. **Hint:** Find $\frac{dy}{dx}$ in terms of n for each of L_1 and L_3. *If there is such an intersecting point*, then $y=-x$ and $\frac{dy}{dx}=1$ at that point.

Q (x, y) is a point on a curve as shown in Graph 5. (Refer to the start of Part II on page 6)

The distance of point Q from the origin O $(0, 0)$ is n times its distance from the line $y = -k$, where $n = \frac{4}{5}$ and $k \neq 1$.

i. Find **an** equation of the curve in Graph 5 in terms of k.

The following regular shape is formed using the curve in Graph 5 as L_1.



Graph 7

j. **Recall:** Plane figures (2-D shapes) are **similar** if a figure is the dilation of another figure in both x and y directions.
Discuss whether the regular shape in Graph 7 is similar to the regular shape in Graph 6.

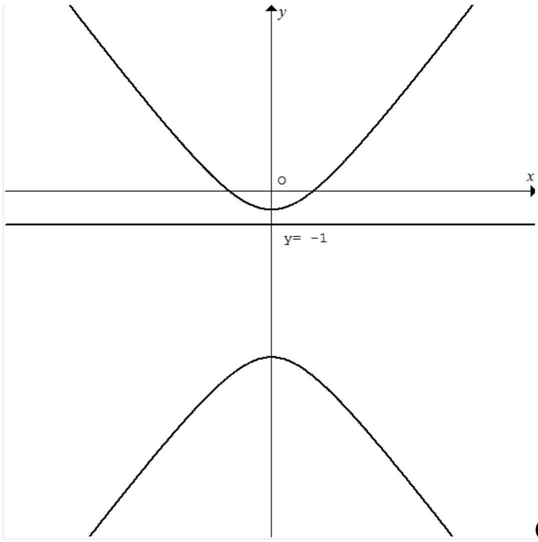
End of Part II

Part III (70-80 min)

$P(x, y)$ is a point on a curve.

The distance of point P from the origin $O(0, 0)$ is n times its distance from the line $y = -1$, where $n > 1$.

Graph 8 shows the curve for $n = \frac{5}{4}$.



Graph 8

a. Use the above information to write an equation, hence show that the equation of the curve can be written as $\frac{x^2}{\left(\frac{5}{3}\right)^2} - \frac{\left(y + \frac{25}{9}\right)^2}{\left(\frac{20}{9}\right)^2} = -1$.

b. Name the curve in part a. Find the exact y-intercepts and coordinates of the centre of the curve.

c. Exactly the same curve $\frac{x^2}{\left(\frac{5}{3}\right)^2}-\frac{\left(y+\frac{25}{9}\right)^2}{\left(\frac{20}{9}\right)^2}=-1$ can also be obtained by repeating part a with the following changes: $\mathrm{O}\,(0,0)$ is changed to $\mathrm{F}\,(a,b)$, line $y=-1$ is changed to $y=c$.

Explain why $a=0$, $b=-\frac{50}{9}$ and $c=-\frac{41}{9}$.

d. Let the coordinates of P be $(4,3)$. $\overline{\mathrm{OP}}$ is the distance of P from O and $\overline{\mathrm{FP}}$ is the distance of P from F.

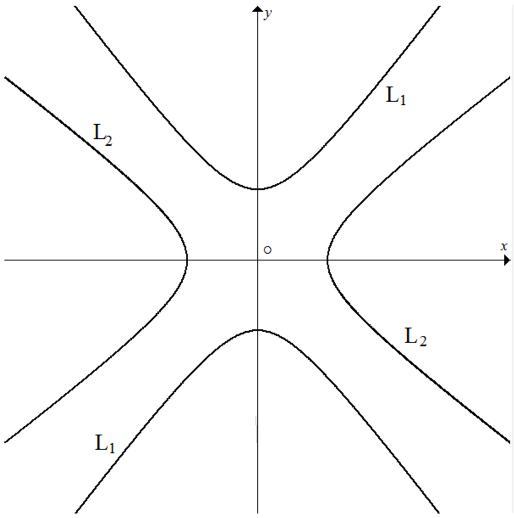
(i) Calculate the exact value of $\left|\overline{\mathrm{FP}}-\overline{\mathrm{OP}}\right|$.

(ii) $\mathrm{P}\,(x,\,y)$ is any point on the curve. Show that $\left|\overline{\mathrm{FP}}-\overline{\mathrm{OP}}\right|$ has the same value as in d(i).

(iii) Write a brief statement to summarise your findings in parts d (i) and d (ii).

e. Write down the equation of the curve such that the origin $\mathrm{O}\,(0,0)$ is the centre of the two branches of the curve.

The curve (two branches) in part e is shown as L_1 in Graph 9 below.

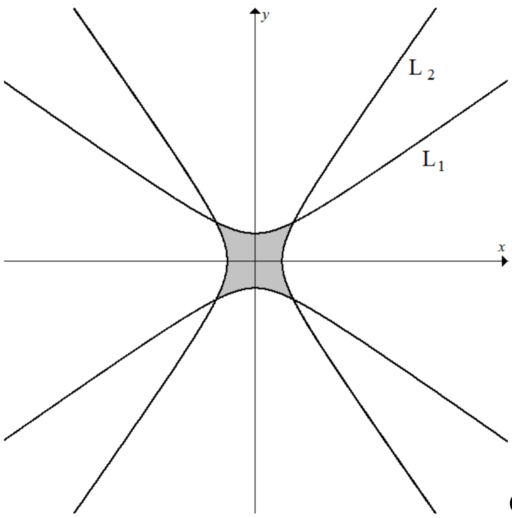


Graph 9

f. L_2 can be obtained by a rotation of L_1 about O by $90°$ in either direction. Name a different transformation of L_1 that will give L_2.

g. Use any methods to find the equation of L_2.

The two curves L_1 and L_2 in Graph 9 do not form a closed region. By changing the value of n, it is possible for L_1 and L_2 to intersect and form a closed region (shaded) as shown in Graph 10 below.



Graph 10

h. Find the range of values of n such that L_1 and L_2 intersect and form a closed region.
Suggestion: Find the equation of the curve in terms of n such that the distance of point P from the origin O $(0, 0)$ is n times its distance from the line $y = -1$, where $n > 1$. Find the gradients of its asymptotes.

End of Task