2023 Specialist Mathematics

Year 12 Modelling Task

(Time allowed: 2.0 hours plus)

Modelling Task

Theme: Catenary

Assumed knowledge: Functions and graphs, algebra, similarity, derivatives, implicit differentiation, length of a curve, area under a curve, parametric equations, use of CAS

Part I (80 minutes plus)

Consider the following graph of equation $y = f(x)$.

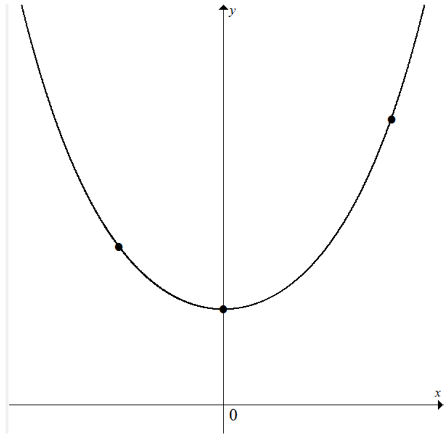

Figure 1

Three points are marked on the graph: $\left(-\log_e 3,\ \frac{5}{3}\right)$, $(0, 1)$ and $\left(\log_e(3+2\sqrt{2}),\ 3\right)$.

a. Show/explain why $f(x)$ is NOT a quadratic function of the form $ax^2 + 1$ where $a \in R^+$ is a constant.

The equation of the above graph is in fact of the form $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$ where $a \in R^+$ is a constant.

The curve is known as **catenary**.

b. Determine the value of a in $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$.

Hence verify that $\left(-\log_e 3,\ \frac{5}{3}\right)$ and $\left(\log_e(3+2\sqrt{2}),\ 3\right)$ are on the curve.

c. Two curves are similar if one curve is the dilation by the same factor in both x and y directions of the other curve.

Explain why curve $y = 3\left(\frac{e^{\left(\frac{x}{3}\right)} + e^{-\left(\frac{x}{3}\right)}}{2}\right)$ is similar to curve $y = e^{\left(\frac{x}{2}\right)} + e^{-\left(\frac{x}{2}\right)}$.

d. Investigate and **comment** on the effects of changing the value of a on the graph of $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$.

Suggestion: Select several appropriate values of a and sketch the graph of each (labelled with value of a) on the same axes.

e. Discuss the effect on the graph in **Figure 1** if $a \in R^{-}$ instead of $a \in R^{+}$.

f. Show that x in the equation $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$ can be expressed as $x = a\log_e\left(\frac{y + \sqrt{y^2 - a^2}}{a}\right)$.

g. From $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$, show that $\frac{dy}{dx} = \frac{e^{\left(\frac{x}{a}\right)} - e^{-\left(\frac{x}{a}\right)}}{2}$.

h. From $x = a\log_e\left(\frac{y + \sqrt{y^2 - a^2}}{a}\right)$, show that $\frac{dx}{dy} = \frac{a}{\sqrt{y^2 - a^2}}$.

i. From part g and part h, show that $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.

j. Find the value of k such that $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$ is a solution to differential equation $\frac{d^2y}{dx^2} - k^2 y = 0$.

k. In terms of a, find the length of curve $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$ from $x = 0$ to $x = a$.

Compare this length with the length of curve $y = \frac{e^x + e^{-x}}{2}$ from $x = 0$ to $x = 1$ and comment.

End of Part I

Part II (80 minutes plus)

In Part II you use some of the findings in Part I to do the following tasks.

An example of a catenary is a chain supported freely at its ends as shown in **Figure 2** below.

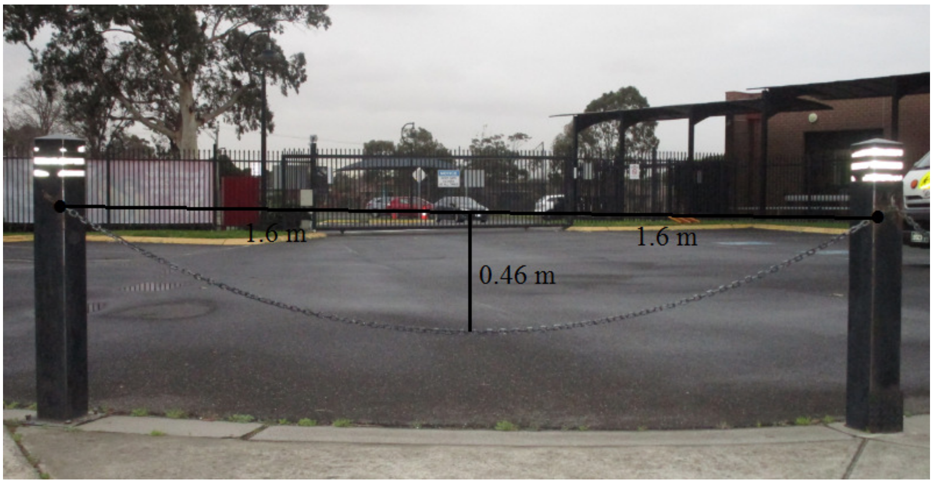

Figure 2

The two ends (shown as dots) of the chain are at the same level and 1.6 m from the midpoint between them. The lowest point of the chain is 0.46 m below the midpoint.

Set the vertical y-axis passing through the midpoint.
The exact location of the x-axis is unknown at this stage, but it is below the chain.

The freely supported chain can be modelled by $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$.

a. State the domain of $f(x) = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$.

b. In terms of a write down the coordinates of the lowest point of the chain.

c. In terms of a write down the y-coordinate of the chain end at $x = 1.6$ using $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$.

d. Explain why $a + 0.46$ is another expression for the y-coordinate of the chain end at $x = 1.6$.

The value of a is approximately 3.

e. Determine the value of a correct to 4 decimal places.

f. Sketch the graph of your model of the freely supported chain using the value of a found in part e.
Use the same scale for both axes in your sketch.
Note: If you are unsure of the value of a, use $a = 3$.

g. Find the acute angle between the chain and the vertical support at the right end of the suspended chain.
Note: If you are unsure of the value of a, use $a = 3$.

h. Find the length of the chain from end to end.
Note: If you are unsure of the value of a, use $a = 3$.

Catenary $y = a\left(\frac{e^{\left(\frac{x}{a}\right)} + e^{-\left(\frac{x}{a}\right)}}{2}\right)$ can be used as a parametric equation for x in a hyperbola of the type $x^2 - y^2 = a^2$.

Let t be the parameter for x and y in $x^2 - y^2 = a^2$ such that $x(t) = a\left(\frac{e^{\left(\frac{t}{a}\right)} + e^{-\left(\frac{t}{a}\right)}}{2}\right)$.

i. Show that the y-coordinate in $x^2 - y^2 = a^2$ is $y(t) = a\left(\frac{e^{\left(\frac{t}{a}\right)} - e^{-\left(\frac{t}{a}\right)}}{2}\right)$.

j. Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Hence find $\frac{dy}{dx}$

k. From $x^2 - y^2 = a^2$, find $\frac{dy}{dx}$. Compare with your answer in part j.

l. The length of hyperbola $x^2 - y^2 = a^2$ from $\left(2a, \sqrt{3}\,a\right)$ to $\left(3a, 2\sqrt{2}\,a\right)$ is λa.
Find the value of λ correct to 4 decimal places.

Figure 3 below shows the graph of $x^2 - y^2 = 1$ for $a = 1$.
$y = \pm x$ are the asymptotes of the curve.

Point $P(x, y)$ is on the curve where $x = \dfrac{e^t + e^{-t}}{2}$ and $y = \dfrac{e^t - e^{-t}}{2}$.

OP is a line segment joining origin O and point $P(x, y)$.

The region bounded by line segment OP, the x-axis and the hyperbola $x^2 - y^2 = 1$ is shaded in **Figure 3**.

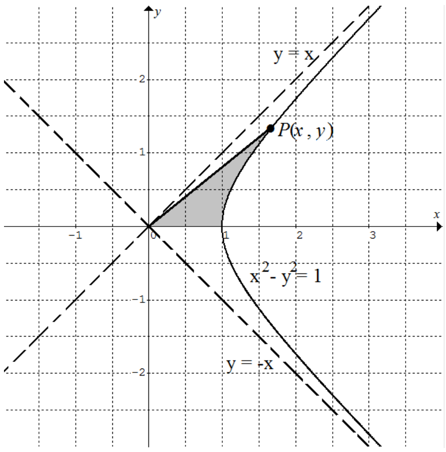

Figure 3

l. Show that point $P(x, y)$ is on the hyperbola $x^2 - y^2 = 1$.

Use $\int \sqrt{x^2 - 1}\, dx = \dfrac{1}{2} x\sqrt{x^2 - 1} - \dfrac{1}{2}\log_e\left|x + \sqrt{x^2 - 1}\right| + C$ for the next parts.

m. Write an expression in finding the area of the shaded region.
Show that the area of the shaded region is $\dfrac{t}{2}$.

n. Hence state what parameter t in $x = \dfrac{e^t + e^{-t}}{2}$ and $y = \dfrac{e^t - e^{-t}}{2}$ represents.

End of Part II