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2023 Specialist Mathematics

Year 12 Problem Solving Task

(Time allowed: 2.0 hours plus)

Problem Solving Task

Theme: Sliding bead

In this task you are to investigate the motion of a bead sliding on a string.

Assumed knowledge:

Newton's laws of motion, static equilibrium, Lami's theorem, mechanics, friction, functions, relations and graphs, calculus, differential equations, use of CAS

Specifications: ***Force*** *is measured in newtons (N),* ***distance (length)*** *in metres (m),* ***time*** *in seconds (s),* ***mass*** *in kilograms (kg) and* ***angle*** *in radians.*

Assumptions:
(1) The bead is a dimensionless particle.
(2) The string is dimensionless and has no mass. The string is not stretchable.
(3) Motion is frictionless and without air resistance unless stated otherwise.
(4) g is $9.8\,N\,kg^{-1}$.

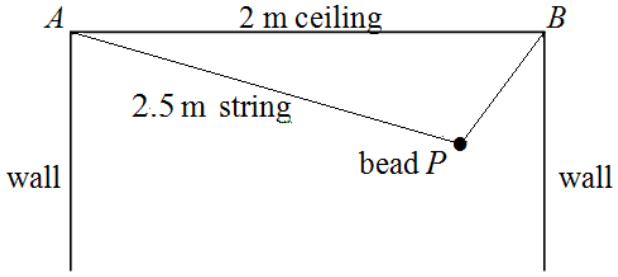


Figure 1

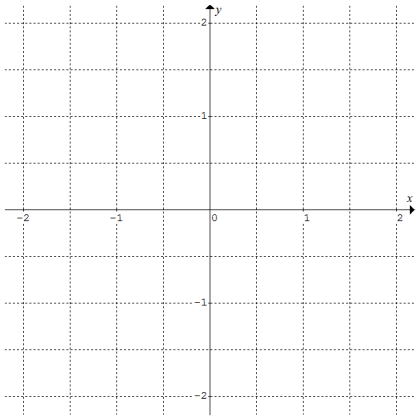
A 2.5 m long string is fastened to corners *A* and *B* of a 2 m wide ceiling.
The bead is at position *P* and it is free to slide along the string between the two walls as shown in **Figure 1**.
Let the *x*-axis run along the ceiling and the *y*-axis pass through the midpoint of *AB*.

Useful information: The bead traces an elliptical path. PA and PB make the same angle with the normal to the path at position P.

Part I (70 – 85 minutes)

a. Show that $\frac{x^2}{25}+\frac{y^2}{9}=\frac{1}{16}$ is the equation of the elliptical path when the bead slides along the string from wall to wall.

b. Sketch the **path** of the bead. Label the featured points with their coordinates.



c. Show that $\frac{dy}{dx} = -\frac{9x}{25y}$ and express $\frac{dy}{dx}$ in terms of x only.

d. Show that $PB = \frac{5-\sqrt{7}}{4}$ when sections PA and PB are perpendicular.

The mass of the bead is $0.01\,\text{kg}$.
For the next five questions (part e to part i) consider there is friction between the bead and the string. Friction prevents the bead from sliding.

e. The bead is at rest (due to friction) and sections PA and PB are perpendicular.
Draw a diagram showing perpendicular sections PA and PB, and use arrows to show forces on the bead.

f. Find the tension (exact value) in each of sections PA and PB when they are perpendicular.

g. When sections PA and PB are perpendicular, show that the x-coordinate of position P is $x = 1 - \frac{1}{2}\left(\frac{5-\sqrt{7}}{4}\right)^2 \approx 0.8268$ and $\frac{dy}{dx} \approx 0.5292$.

h. Calculate the force of friction preventing the bead from sliding.

Select a suitable value for the length of section PB where $0.7 \leq PB \leq 1.1$ to specify a new position.

i. The bead is at rest (due to friction) at the new position. Find the tension in each of sections PA and PB.

j. If there is **no** friction between the bead and the string (i.e. the bead slides freely along the string), find the tension in the string when the bead is at the new position as in part i for your selected value of PB.
Hint: Same tension in both sections.

k. If there is no friction between the bead and the string, discuss/explain whether the tension in the string depends on the position of the bead, without actually doing calculations.

End of Part I

Part II (60 – 75 minutes)

Specifications: ***Force*** *is measured in newtons (N),* ***distance (length)*** *in metres (m),* ***time*** *in seconds (s),* ***mass*** *in kilograms (kg) and* ***angle*** *in radians.*

Assumptions:
(1) The bead is a dimensionless particle.
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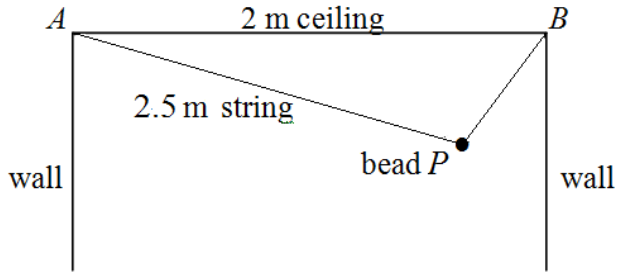

Figure 1

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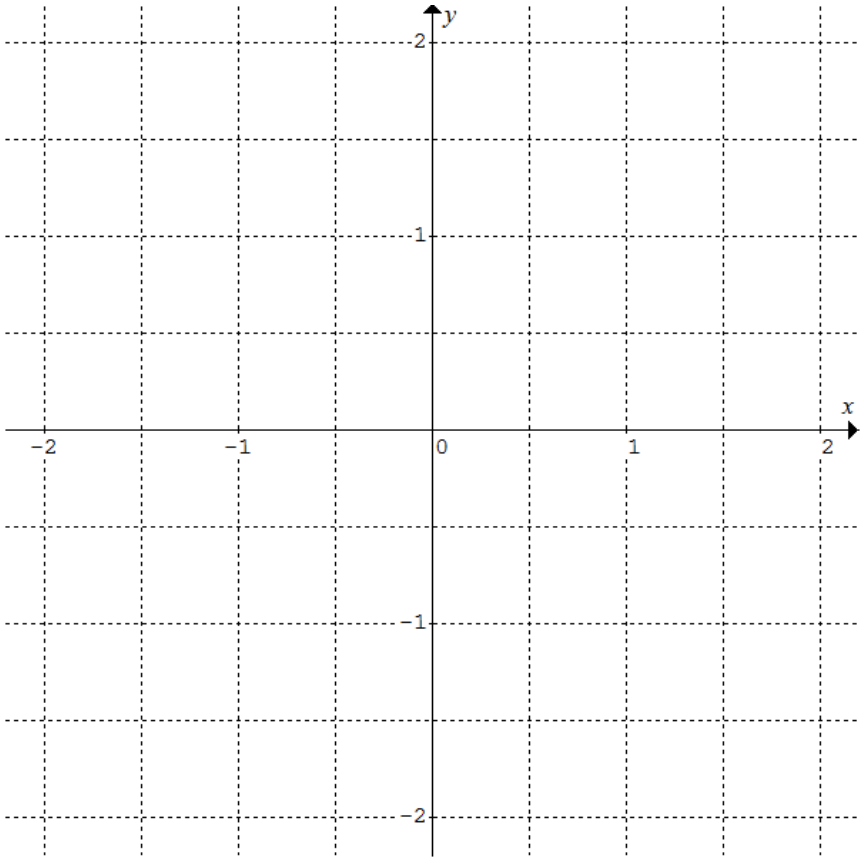
The bead is at position *P* and it is free to slide along the string between the two walls as shown in **Figure 1**.
Let the *x*-axis run along the ceiling and the *y*-axis pass through the midpoint of *AB*.

Useful information: The bead traces an elliptical path of equation $\frac{x^2}{25}+\frac{y^2}{9}=\frac{1}{16}$.
PA and PB make the same angle with the normal to the path at position P.

The mass of the bead is 0.01 kg.

a. Consider the bead at $P(x, y)$ where $x \in (0.6, 0.9)$. There is **no** friction between the bead and the string.
Draw a diagram showing the bead and its path, and forces (use arrows) acting on the bead.
Use dotted lines to represent the tangent and the normal to the path at $P(x, y)$.



b. Show that the angle θ between the vertical and the normal to the path at $P(x, y)$ is given by $\tan\theta = \dfrac{12x}{5\sqrt{25-16x^2}}$.

c. Show that $\sin\theta = \dfrac{12x}{\sqrt{625-256x^2}}$ and $\cos\theta = \dfrac{5\sqrt{25-16x^2}}{\sqrt{625-256x^2}}$.

d. Calculate the magnitude of the resultant force of the string on the bead in terms of x.

e. Calculate the component of the force of gravity on the bead in the direction of the tangent line at $P(x, y)$ in terms of x.

f. Show that the acceleration of the bead at $P(x, y)$ is $a = -\dfrac{117.6x}{\sqrt{625 - 256x^2}}$ in the direction of the tangent.

g. Select three appropriate small values of $x \in (-0.2, 0.2)$, show that $a \approx -4.7x$.

h. Write a differential equation involving x and time t for small values of $x \in (-0.2, 0.2)$.

i. Show that $x = 0.1\sin\left(\sqrt{4.7}\, t\right)$ is a solution to the differential equation in part h.

j. Write down two other solutions to the differential equation in part h.

k. Determine the period of the motion represented by the differential equation for small values of x.

End of Part II End of Problem Solving Task