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2023 VCAA Mathematical Methods Exam 1 Solutions

Q1a $\dfrac{dy}{dx}=\dfrac{e^x(2x-1)-(x^2-x)e^x}{(e^x)^2}=\dfrac{-x^2+3x-1}{e^x}=-\dfrac{x^2-3x+1}{e^x}$

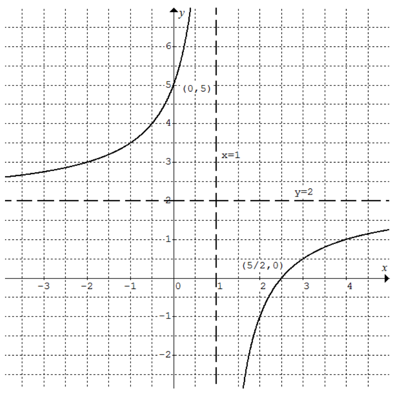
Q1b $f'(x)=e^{2x}\cos x+2e^{2x}\sin x=e^{2x}(\cos x+2\sin x)$

$f'\left(\dfrac{\pi}{4}\right)=e^{\frac{\pi}{2}}\left(\cos\dfrac{\pi}{4}+2\sin\dfrac{\pi}{4}\right)=e^{\frac{\pi}{2}}\left(\dfrac{1}{\sqrt{2}}+\sqrt{2}\right)=\dfrac{3\sqrt{2}}{2}e^{\frac{\pi}{2}}$

Q2 $(e^x)^2-4e^x-12=0$, $(e^x-6)(e^x+2)=0$

Since $e^x+2>0$.: $e^x-6=0$.: $x=\log_e 6$

Q3a



Q3b $f(x)=1$ when $x=4$, $f(x)\le 1$ when $1<x\le 4$

Q4 $y=f(x)=x+\dfrac{1}{x}$, $f(1)=2$, $f(2)=\dfrac{5}{2}$, $f(3)=\dfrac{10}{3}$

Area $\approx\dfrac{1}{2}\left(2+\dfrac{5}{2}\right)\times 1+\dfrac{1}{2}\left(\dfrac{5}{2}+\dfrac{10}{3}\right)\times 1=\dfrac{31}{6}$

Q5a $\displaystyle\int_0^{\frac{\pi}{3}}\sin x\,dx=[-\cos x]_0^{\frac{\pi}{3}}=-\cos\frac{\pi}{3}+\cos 0=-\frac{1}{2}+1=\frac{1}{2}$

Q5b $\displaystyle\int_k^{\frac{\pi}{2}}\cos x\,dx=[\sin x]_k^{\frac{\pi}{2}}=1-\sin k=\frac{1}{2}$ where $-3\pi\le k\le 2\pi$

.: $\sin k=\dfrac{1}{2}$, $k=-\dfrac{11\pi}{6},-\dfrac{7\pi}{6},\dfrac{\pi}{6},\dfrac{5\pi}{6}$

Q6a $\hat{p}=\dfrac{0.04+0.16}{2}=0.10$

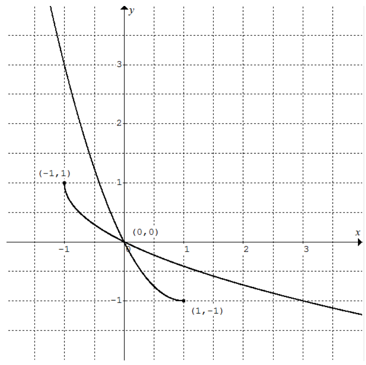
Q6b $\hat{p}+2\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}=0.16$, $0.10+2\sqrt{\dfrac{0.10(1-0.10)}{n}}=0.16$

$\sqrt{\dfrac{0.09}{n}}=0.03$, $\dfrac{0.09}{n}=0.0009$, $n=100$

Q6c Width $\propto\dfrac{1}{\sqrt{n}}$, when n is four times, width is halved.

Q7a $[-1,\infty)$

Q7b



Q7c $x=y^2-2y$ where $y\le 1$ and $x\ge -1$

$y^2-2y-x=0$, $y=\dfrac{2-\sqrt{4+4x}}{2}=1-\sqrt{1+x}$ for $x\in[-1,\infty)$

Q7d Area $\displaystyle=2\times\int_0^1\left(-x-(x^2-2x)\right)dx=2\times\left[-\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1=\frac{1}{3}$

Q8a $\displaystyle k\int_0^4(16t-t^3)\,dt=1$, $\displaystyle k\left[8t^2-\frac{t^4}{4}\right]_0^4=k(128-64)=1$, $k=\dfrac{1}{64}$

Q8b $\displaystyle \mathrm{E}(T)=\int_0^4 tf(t)\,dt=\frac{1}{64}\int_0^4(16t^2-t^4)\,dt=\frac{1}{64}\left[\frac{16}{3}t^3-\frac{t^5}{5}\right]_0^4=\frac{32}{15}$

Q8c $\Pr(T>2\,|\,T>1)=\dfrac{\Pr(T>2\cap T>1)}{\Pr(T>1)}=\dfrac{\Pr(T>2)}{\Pr(T>1)}$

$=\dfrac{k\left[8t^2-\frac{t^4}{4}\right]_2^4}{k\left[8t^2-\frac{t^4}{4}\right]_1^4}=\dfrac{\left(8(4^2)-\frac{4^4}{4}\right)-\left(8(2^2)-\frac{2^4}{4}\right)}{\left(8(4^2)-\frac{4^4}{4}\right)-\left(8(1^2)-\frac{1}{4}\right)}=\dfrac{16}{25}$

Q9a $f(0)=a-0=a=12$

$g(1)=12+b=9$.: $b=-3$

Q9b $f(x)=12-x(x-2)^2$

$f'(x)=-(x-2)(3x-2)=0$, $x=\dfrac{2}{3}$ or 2

Given the shape of track 1, P is at $x=2$.: $y=f(2)=12$

$g(x)=12x-3x^2=3x(4-x)$

.: turning point is on the axis of symmetry $x=2$, $y=g(2)=12$

.: both have the same turning point $P(2,12)$.

Q9c Area $A(k)=\dfrac{1}{2}k\times g(k)=\dfrac{1}{2}k(12k-3k^2)=6k^2-\dfrac{3}{2}k^3$, $k\in(0,4)$

Let $\dfrac{dA}{dk}=0$.: $12k-\dfrac{9}{2}k^2=3k\left(4-\dfrac{3}{2}k\right)=0$.: $k=\dfrac{8}{3}$

Max area $=A\left(\dfrac{8}{3}\right)=\dfrac{128}{9}$ km^2

Please inform mathline@itute.com re conceptual and/or mathematical errors