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2023 VCAA Mathematical Methods Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
E	A	E	B	D	D	C	C	C	B

11	12	13	14	15	16	17	18	19	20
E	E	C	D	A	B	B	E	D	A

Q1 Amplide $=\left|-\frac{1}{2}\right|=\frac{1}{2}$, period $=\frac{2\pi}{3}$ E

Q2 Axis of symmetry $x=\frac{-2b}{2a}=-\frac{b}{a}$ A

Q3 $[-2,3)\cap(-1,5]=(-1,3)$ E

Q4 $\frac{k}{5}x+y=\frac{k}{5}+1$ and $\frac{4}{k+1}x+y=0$

Infinite solutions when $\frac{k}{5}+1=0$ and $\frac{k}{5}=\frac{4}{k+1}$

$k=-5$ satisfies both equation B

Q5 Index >1 D

Q6 $\int_7^3 f(x)dx=-\int_3^7 f(x)dx=-(C-D)=D-C$ D

Q7 $\frac{d}{dx}(f\circ g)(x)=\frac{1}{2(x-1)}$ where $x<1$ C

Q8 $\Pr(X\geq 1)=1-\Pr(X=0)=1-\left(\frac{m}{n+m}\right)^8$ C

Q9 Let $\frac{d}{dx}\tan\frac{x}{2}=\frac{d}{dx}\sin ax$ at $x=2\pi$, $a=-\frac{1}{2}$ C

Q10 $k\in[1,6)$, $\frac{1}{2}(k-1)\times\frac{(k-1)}{20}=0.35$, $k=\sqrt{14}+1$ B

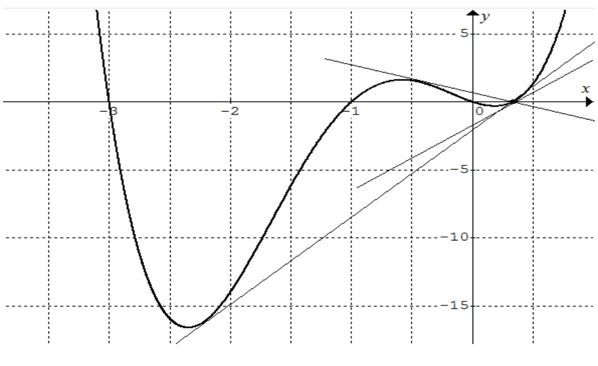
Q11 $\frac{d}{dx}f(x)\times g(x)=f'(x)g(x)+f(x)g'(x)$

At $x=-2$, $f'(-2)g(-2)+f(-2)g'(-2)=10$ E

Q12 $k\geq 0$, mean of X has a max value of 2 at $k=0$ E

Q13 First 0.83333, second 0.81785, third 0.81773 C

Q14 D



Q15 A

Q16 $g(x)=\frac{e^{(x-1)^2}}{e^{(x-1)}}=e^{(x-1)^2-(x-1)}=e^{(x-2)(x-1)}$ B

Q17 $r=\frac{y}{2\pi}$, $h=x-4r=x-\frac{2y}{\pi}$, $V=\pi r^2h=\frac{\pi xy^2-2y^3}{4\pi^2}$ B

Q18 $\sin x$ has 1 local minimum in $[-\pi,\pi]$, $\sin ax$ has a local minima in $[-\pi,\pi]$, $\sin ax$ has $a\times a$ local minima in $[-a\pi,a\pi]$ E

Q19 $\Delta=6(4k+3)>0$.: $4k+3>0$ and $k>-\frac{3}{4}$ for two solutions

$x=\frac{-(4k+3)\pm\sqrt{\Delta}}{2}$, for a positive solution $-(4k+3)+\sqrt{\Delta}>0$

.: $k<\frac{3}{4}$.: $-\frac{3}{4}<k<\frac{3}{4}$ D

Q20 For $f\circ g(x)$ to be defined, $g\in\left(-\frac{1}{\sqrt{2}},\infty\right)\cap[-1,1]=\left(-\frac{1}{\sqrt{2}},1\right]$

.: $x\in\left(-\frac{\pi}{4},\frac{5\pi}{4}\right)\cup\left(-\frac{9\pi}{4},-\frac{3\pi}{4}\right)\cup\cdots$ Interval A

For $g\circ f(x)$ to be defined, $f\in(-\infty,5)$

.: $x\in\left(-\frac{1}{\sqrt{2}},e^5-\frac{1}{\sqrt{2}}\right)$ Interval B

For both to exist, $x\in A\cap B=\left(-\frac{1}{\sqrt{2}},\frac{5\pi}{4}\right)$. A

SECTION B

Q1a $(-1,0),(0,0),(2,0)$

Q1b Let $f'(x)=0$, $x=\frac{1-\sqrt{7}}{3}$, $x=\frac{1+\sqrt{7}}{3}$ and the corresponding

y-coordinates $y=\frac{2(7\sqrt{7}-10)}{27}$, $y=\frac{-2(7\sqrt{7}+10)}{27}$

Turning points are $\left(\frac{1-\sqrt{7}}{3},\frac{2(7\sqrt{7}-10)}{27}\right)$, $\left(\frac{1+\sqrt{7}}{3},\frac{-2(7\sqrt{7}+10)}{27}\right)$

Q1ci $x(x-2)(x+1)=(x-2)$, $x(x-2)(x+1)-(x-2)=0$

$(x-2)(x(x+1)-1)=0$, $(x-2)(x^2+x-1)=0$

.: $x=2$ or $x=\frac{-1\pm\sqrt{5}}{2}$

Q1cii Area of the bounded region

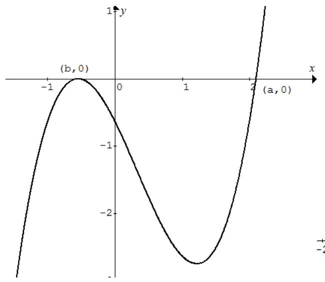
$=\int_a^b(x-2)(x^2+x-1)\,dx+\int_b^2-(x-2)(x^2+x-1)\,dx$

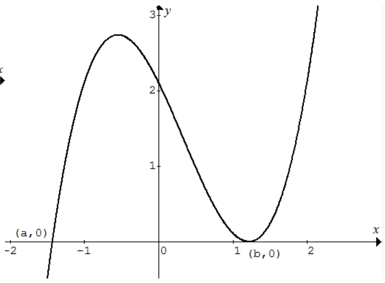
where $a=\frac{-1-\sqrt{5}}{2}$ and $b=\frac{-1+\sqrt{5}}{2}$

Q1ciii Total area ≈ 5.95

Q1d Square factor indicates turning point on the x-axis.

Two possibilities: One by lowering $f(x)$ by $\frac{2(7\sqrt{7}-10)}{27}$, the other by raising $f(x)$ by $\frac{2(7\sqrt{7}+10)}{27}$





Equations:

$y=f(x)-\frac{2(7\sqrt{7}-10)}{27}$ $\qquad$ $y=f(x)+\frac{2(7\sqrt{7}+10)}{27}$

x-intercepts:

$b=\frac{1-\sqrt{7}}{3}$, $a=\frac{1+2\sqrt{7}}{3}$ $\qquad$ $a=\frac{-1-2\sqrt{7}}{3}$, $b=\frac{1+\sqrt{7}}{3}$

Q2a Period $=\frac{2\pi}{b}=30$, $b=\frac{\pi}{15}$. $h(0)=-60\cos 0+c=15$, $c=75$

Q2b Average height $=\frac{\int_0^{7.5}\left(-60\cos\left(\frac{\pi t}{15}\right)+75\right)dt}{7.5}\approx 36.80$

Q2c Average of change in height $=\frac{75-15}{7.5}=8$

Q2di $k=75+60=135$, period $=\frac{2\pi}{bm}=\frac{30}{2}$.: $m=2$

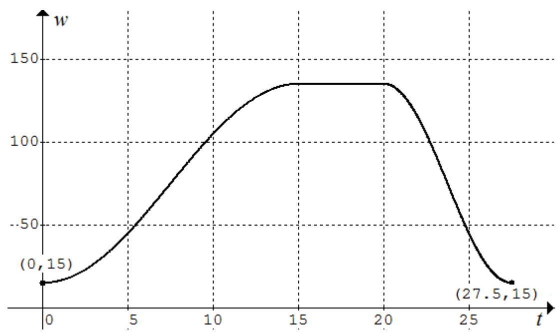
Q2dii Half of a turn in the last 7.5 minutes, returns to point A.

$h(27.5)=-60\cos\left(\frac{\pi}{15}(2\times 27.5+n)\right)+75=15$, .: $\cos\left(\frac{\pi}{15}(55+n)\right)=1$

.: $\frac{\pi}{15}(55+n)=2\alpha\pi$.: $n=30\alpha-55$ or $n=30\alpha+5$ where $\alpha\in Z$

(Note: Can use $h(20)$ to find other equivalent forms of n)

Q2diii



Q3a As $x\to-\infty$, $2^x+5\to 5$

Q3b $g(x)=2^x+5=\left(e^{\log_e 2}\right)^x=e^{x\log_e 2}+5$

$g'(x)=\log_e 2\times e^{x\log_e 2}=\log_e 2\times 2^x$.: $k=\log_e 2$

Q3ci Tangent: $y-\left(2^a+5\right)=\log_e 2\times 2^a\,(x-a)$, simplify to

$y=\log_e 2\times 2^a\,x-2^a a\log_e 2+2^a+5$ at $(a, g(a))$

Q3cii Tangent passes through $(0,0)$, $-2^a a\log_e 2+2^a+5=0$ and

$a\approx 2.61785$.: $\log_e 2\times 2^a\approx 4.25478$ Hence $y\approx 4.255x$

Q3d Let $h''(x)=0$.: $x\approx 2.05753$ and $y\approx h(2.05753)\approx -0.0707$

Point of inflection $(2.06,-0.07)$, correct to 2 decimal places

Q3e The interval between the two turning points, $[0.49, 3.21]$, correct to 2 decimal places

Q3f

x_0	0
x_1	- 1.443
x_2	- 0.897
x_3	- 0.773

Q3g $\log_e(2)\times\left(2^x\right)-2x=h'(x)$ is a denominator in Newton's method formula. It makes the calculation undefined when $h'(x)=0$

.: a solution to $h'(x)=0$ should not be used as x_0.

Q3h Local minimum on the x-axis: At the local minimum

$f(x)=n^x-x^n=0$ and $f'(x)=(\log_e n)n^x-nx^{n-1}=0$

A solution to the first equation is $x=n$ by inspection

.: $f'(n)=(\log_e n)n^n-nn^{n-1}=(\log_e n-1)n^n=0$

.: $\log_e n=1$.: $n=e$

Q4a Normal: $\mu=6.7$, $\sigma=0.1$ $\Pr(D>6.8)\approx).1587$

Q4b $\Pr(D<d)=0.90$, $\Pr\left(Z<\frac{d-6.7}{0.1}\right)=0.90$, $d\approx 6.83$

Q4c $\Pr(D<6.95)\approx 0.9938$ $\quad$ (0.99379)

Q4d Binomial: $n=4$, $p\approx 0.993790$, random variable X is the number of balls. $\Pr(X\geq 3)\approx 0.9998$

Q4e $\Pr(A\,|\,\text{fit})=\frac{\Pr((6.54<D<6.86)\cap(D<6.95))}{\Pr(D<6.95)}$

$=\frac{\Pr(6.54<D<6.86)}{\Pr(D<6.95)}\approx 0.8960$

Q4f The mean 6.7 is the mid point between 6.54 and 6.86.
Probability of a ball outside the interval is $1-0.99=0.01$

.: require $\Pr(D<6.54)<0.005$, $\Pr\left(Z<\frac{6.54-6.7}{\sigma}\right)<0.005$

$\frac{6.54-6.7}{\sigma}<-2.5758$, $\sigma<0.0621\approx 0.06$

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Q4g $n = 32$, $\mu = 6.7$, $\sigma = 0.1$, confidence interval $(0.7382,\ 0.9493)$

$$\mathrm{E}(\hat{P}) = p = \frac{0.7382 + 0.9493}{2} = 0.84375$$

$$\mathrm{sd}(\hat{P}) = \sqrt{\frac{0.84375(1 - 0.84375)}{32}} \approx 0.064186$$

$0.9493 = 0.84375 + z \times 0.064186 \quad \therefore\ z \approx 1.64444$

$\Pr(-1.64444 < Z < 1.64444) \approx 0.8999 \approx 90\%$

Q4h $\displaystyle\int_{50}^{3\pi^2+30} \frac{1}{6\pi} \sin\sqrt{\frac{v-30}{3}}\, dv \approx 0.1345$

Q4i $\displaystyle\int_{30}^{3\pi^2+30} \frac{1}{6\pi} v \sin\sqrt{\frac{v-30}{3}}\, dv = 3\pi^2 + 12$

Q4j When $f(v)$ is dilated in both directions, the enclosed area remains as 1 $\therefore\ ab = 1$

The mean of $g(w)$ $= \displaystyle\int_{30b}^{b(3\pi^2+30)} wg(w)\, dw = ab^2 \int_{30}^{3\pi^2+30} vf(v)\, dv$

$\therefore\ 2\pi^2 + 8 = ab^2(3\pi^2 + 12)$ $\therefore\ ab^2 = \dfrac{2}{3}$ and $ab = 1$

$\therefore\ b = \dfrac{2}{3}$ and $a = \dfrac{3}{2}$

Q5a

- Reflection in the *y*-axis
- Translation of 2 units in the positive *x*-direction

Alternatively:

- Translation of 2 units in the negative *x*-direction
- Reflection in the *y*-axis

Q5b $f(x)$ has a turning point at $(2, 1)$

Assume that 'create' means splitting $f(x)$ into two.

Create the two functions by splitting $f(x)$ at its turning point.

A possible split: g_1 has domain $[2, \infty)$ and range $[1, \infty)$.

g_1^{-1} has domain $[1, \infty)$ and range $[2, \infty)$.

Alternatively:

g_1 has domain $(2, \infty)$ and range $(1, \infty)$.

g_1^{-1} has domain $(1, \infty)$ and range $(2, \infty)$.

Without the assumption, there are in fact infinitely many possibilities. The question should include the word 'maximal' to eliminate all but the first possibility above.

Q5ci $P(1.27, 1.27)$, $Q = (4.09, 4.09)$

Q5cii $\text{Area} = 2 \times \displaystyle\int_{1.2747}^{4.0852} \left(x - \frac{1}{2}\left(e^{2-x} + e^{x-2}\right) \right) dx \approx 5.56$

Q5d Turning point of $f(x)$ is $(0, 2)$.

$h(x)$ is the transformation of $f(x)$

$\therefore$ the turning point of $h(x)$ is $\left(k, \dfrac{2}{k}\right)$

At the turning points, the *x* and *y* coordinates have the relationship

$y = \dfrac{2}{x}$ i.e. $y = 2x^{-1}$ $\therefore\ n = -1$

Q5e The smallest k occurs when the inverse of h_1 curve and the h curve touch each other and the line $y = x$.

$\therefore\ h(x) = \dfrac{1}{k}\left(e^{k-x} + e^{x-k}\right) = x$ and $h'(x) = \dfrac{1}{k}\left(-e^{k-x} + e^{x-k}\right) = 1$ at the intersection, where $k > 0$.

Graph both relations to find the intersection or solve them as simultaneous equations, $k \approx 1.2687 \approx 1.27$

Q5f $k = 5$, $h_1^{-1}(x) = \log_e\left(\dfrac{5}{2}x + \dfrac{1}{2}\sqrt{25x^2 - 4}\right) + 5$ and

$h(x) = \dfrac{1}{5}\left(e^{5-x} + e^{x-5}\right)$. The two functions intersect at $x \approx 1.4509,\ 8.7816$

$\text{Area} \approx \displaystyle\int_{1.4509}^{8.7816} \left(\log_e\left(\frac{5}{2}x + \frac{1}{2}\sqrt{25x^2 - 4}\right) + 5 - \frac{1}{5}\left(e^{5-x} + e^{x-5}\right) \right) dx \approx 43.91$

Please inform admin@itute.com re conceptual and/or mathematical errors