2023 VCAA Specialist Mathematics Exam 1 Sample questions Solutions

Q1a If $n=1$, $\frac{1}{2}=1-\frac{1}{2^1}$ is true

Q1b Assumption: For $n=k$,
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}=1-\frac{1}{2^k}$

Q1c For $n=k+1$,
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2}\times\frac{1}{2^k}$
$=1-\frac{1}{2}\times\frac{1}{2^k}=1-\frac{1}{2^{k+1}}$
.: the statement $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}=1-\frac{1}{2^n}$ is true for $n=k+1$.: the statement is true $\forall n\in N$

Q2a For $n=n_0=5$, $2^5=32$, $5^2=25$.: $2^5>5^2$ is true

Q2b From part a, the statement $2^n>n^2$ is true for $n=5$.
Assumption: For $n=k>5$, $2^k>k^2$
Note that $k>2+\frac{1}{k}$.: $k^2>2k+1$.: $2k^2>k^2+2k+1$
.: $2k^2>(k+1)^2$
For $n=k+1$ where $k>5$, $2^{k+1}=2\times 2^k>2k^2>(k+1)^2$
.: the statement $2^n>n^2$ is true for $n=k+1$
.: the statement $2^n>n^2$ is true for $\forall n\geq 5$

Q3 Statement: 9^n-5^n is divisible by 4 for $\forall n\in N$
For $n=1$, $9^1-5^1=4$.: 9^1-5^1 is divisible by 4
Assumption: For $n=k$, 9^k-5^k is divisible by 4
For $n=k+1$,
$9^{k+1}-5^{k+1}=9\times 9^k-5\times 5^k=5\times 9^k-5\times 5^k+4\times 9^k$
$=5\left(9^k-5^k\right)+4\times 9^k$
Since both terms are divisible by 4 .: $9^{k+1}-5^{k+1}$ is divisible by 4
.: the statement is true for $n=k+1$
.: the statement is true for $\forall n\in N$

Q4 If n is odd and assume n^3+1 is odd, the assumption results in n^3 being even and hence n is even which is a contradiction to n is odd.
.: The assumption n^3+1 is odd must be wrong
.: n^3+1 is even.

Q5 Assume that $\sqrt{3}+\sqrt{5}\leq\sqrt{11}$
Squaring both sides: $8+2\sqrt{15}\leq 11$.: $\sqrt{15}\leq\frac{3}{2}$.: $15\leq\frac{9}{4}$
which is false .: $\sqrt{3}+\sqrt{5}>\sqrt{11}$

Q6 $y=\sqrt{4-x^2}$, $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\frac{2}{\sqrt{4-x^2}}\,dx$
$A=2\times\int_0^1 2\pi y\,ds=2\times\int_0^1 4\pi\,dx=8\pi[x]_0^1=8\pi$

Q7 $y=\sqrt[3]{x}$, $x=y^3$, $\frac{dx}{dy}=3y^2$, $(0,0)$, $(8,2)$
$ds=\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\sqrt{1+9y^4}\,dy$
$A=\int_{x=0}^{x=8}2\pi x\,ds=2\pi\int_0^2 y^3\sqrt{1+9y^4}\,dy$
Let $u=1+9y^4$, $\frac{1}{36}du=y^3dy$, $0\to 1$, $2\to 145$
$A=\frac{\pi}{18}\int_1^{145}\sqrt{u}\,du=\frac{\pi}{18}\left[\frac{2u^{\frac{3}{2}}}{3}\right]_1^{145}=\frac{\pi}{27}\left(145^{\frac{3}{2}}-1\right)$

Q8 $x=\sin^3\theta$, $y=\cos^3\theta$
$ds=\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=3\sin\theta\cos\theta$
$A=\int_{\theta=0}^{\theta=\frac{\pi}{2}}2\pi\sin^3\theta\,ds=6\pi\int_0^{\frac{\pi}{2}}\sin^4\theta\cos\theta\,d\theta$
Let $u=\sin\theta$, $du=\cos\theta\,d\theta$
$A=6\pi\int_0^1 u^4\,du=\frac{6\pi}{5}$

Q9 $x=\frac{4}{3}(t+1)^{\frac{3}{2}}$, $y=\frac{1}{2}t^2$, $0\leq t\leq 1$
$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{4(t+1)+t^2}\,dt=(t+2)dt$
$A=\int_{t=0}^{t=1}2\pi y\,ds=\pi\int_0^1\left(t^3+2t^2\right)dt=\pi\left[\frac{t^4}{4}+\frac{2t^3}{3}\right]_0^1=\frac{11\pi}{12}$

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Q10a Let $\frac{dP}{dt}=0$, $P=48000$

Q10b Let $\frac{d^2P}{dt^2}=0$, $P=24000$

Q10c $\frac{dP}{dt}=2P\left(6-\frac{P}{8000}\right)=\frac{1}{4000}P(48000-P)$

$t=4000\int\frac{1}{P(48000-P)}dP$ and $P=4000$ at $t=0$

Partial fractions: $t=\frac{1}{12}\int\left(\frac{1}{P}+\frac{1}{48000-P}\right)dP$

$\ln P-\ln(48000-P)+c=12t$ where $c=\ln 11$

$\therefore\ e^{12t}=\frac{11P}{48000-P}$, $P=\frac{48000e^{12t}}{11+e^{12t}}$

Q11 $I=\int x^2\cos(2x)dx$

Integration by parts $\int udv=uv-\int vdu$

Let $u=x^2$ and $\cos(2x)dx=dv$

$\therefore\ du=2xdx$ and $v=\frac{1}{2}\sin(2x)$

$\int x^2\cos(2x)dx=\frac{1}{2}x^2\sin(2x)-\int x\sin(2x)dx$

Integration by parts

$\int x\sin(2x)dx=-\frac{1}{2}x\cos(2x)+\frac{1}{4}\sin(2x)$

$\therefore\ I=\frac{1}{2}x^2\sin(2x)+\frac{1}{2}x\cos(2x)-\frac{1}{4}\sin(2x)+c$

Q12 Vector orthogonal to the plane where

$\tilde{a}=2\tilde{i}-3\tilde{j}+\tilde{k}$ and $\tilde{b}=4\tilde{i}+2\hat{j}-3\tilde{k}$ lie is

$$\tilde{n}=\tilde{a}\times\tilde{b}=\begin{vmatrix}\tilde{i} & \tilde{j} & \tilde{k}\\ 2 & -3 & 1\\ 4 & 2 & -3\end{vmatrix}=7\tilde{i}+10\tilde{j}+16\tilde{k}$$

$\tilde{r}=x\tilde{i}+y\tilde{j}+z\tilde{k}$, $\tilde{r}_0=3\tilde{i}+2\tilde{j}+\tilde{k}$

$\tilde{r}-\tilde{r}_0=(x-3)\tilde{i}+(y-2)\tilde{j}+(z-1)\tilde{k}$

Cartesian equation of the plane is

$n.(r-r_0)=0$, i.e., $7x+10y+16z=57$

Q13a Vector orthogonal to the plane where

$\overrightarrow{PQ}=\tilde{a}=-2\tilde{i}-4\tilde{j}-4\tilde{k}$ and $\overrightarrow{PR}=\tilde{b}=2\tilde{i}-\hat{j}-6\tilde{k}$ lie is

$$\tilde{n}=\tilde{a}\times\tilde{b}=\begin{vmatrix}\tilde{i} & \tilde{j} & \tilde{k}\\ -2 & -4 & -4\\ 2 & -1 & -6\end{vmatrix}=20\tilde{i}-20\tilde{j}+10\tilde{k}$$

$\tilde{r}=x\tilde{i}+y\tilde{j}+z\tilde{k}$, $\tilde{r}_0=3\tilde{i}+3\tilde{j}+6\tilde{k}$

$\tilde{r}-\tilde{r}_0=(x-3)\tilde{i}+(y-3)\tilde{j}+(z-6)\tilde{k}$

Cartesian equation of the plane is

$n.(r-r_0)=0$, i.e., $2x-2y+z=6$

Q13b $\tilde{r}=(2+2t)\tilde{i}-4t\tilde{j}+(5-3t)\tilde{k}$

At $2x-2y+z=6$, $2(2+2t)-2(-4t)+5-3t=6$

$\therefore\ t=-\frac{1}{3}$ $\therefore\ \tilde{r}=\frac{4}{3}\tilde{i}+\frac{4}{3}\tilde{j}+6\tilde{k}$

$\therefore$ the point of intersection is $\left(\frac{4}{3},\frac{4}{3},6\right)$

Q14 Plane $2x+y+z=7$ has $\tilde{n}=2\tilde{i}+\tilde{j}+\tilde{k}$.

A vector on the line is $\tilde{\ell}=\tilde{i}+2\tilde{j}-\tilde{k}$.

Angle between $\tilde{n}$ and $\tilde{\ell}$ is ϕ such that $\cos\phi=\frac{\tilde{n}.\tilde{\ell}}{|\tilde{n}||\tilde{\ell}|}=\frac{1}{2}$

$\therefore\ \phi=\frac{\pi}{3}$ and angle between the plane and $\tilde{\ell}$ is

$\theta=\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}$

Q15a $\overrightarrow{AB}=2\tilde{i}+\tilde{j}-5\tilde{k}$

$\tilde{r}=3\tilde{i}+\tilde{j}-\tilde{k}+t\left(2\tilde{i}+\tilde{j}-5\tilde{k}\right)$,

$\tilde{r}=(3+2t)\tilde{i}+(1+t)\tilde{j}-(1+5t)\tilde{k}$

Q15b $\tilde{n}=\tilde{i}+2\tilde{j}-\tilde{k}$, $\cos\phi=\frac{\tilde{n}.\overrightarrow{AB}}{|\tilde{n}||\overrightarrow{AB}|}=\frac{3}{\sqrt{20}}$

Angle between the line and the plane is $\theta=\frac{\pi}{2}-\phi$.

$\sin\theta=\sin\left(\frac{\pi}{2}-\phi\right)=\cos\phi=\frac{3}{\sqrt{20}}$

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Q16 $\dot{\tilde{r}} = 2t\tilde{i} + 5\tilde{j} + (2t-16)\tilde{k}$

$v^2 = \dot{\tilde{r}}.\dot{\tilde{r}} = 4t^2 + 25 + (2t-16)^2 = 8t^2 - 64t + 281$

Min. speed when $\frac{d}{dt}v^2 = 0$, i.e. when $t = 4$ s

Min. speed $= \sqrt{8\times 4^2 - 64\times 4 + 281} = \sqrt{153}$ m s^{-1}

Q17 $\tilde{n}_1 = \tilde{i} + \tilde{j} - \tilde{k}$, $\tilde{n}_2 = 2\tilde{i} - \tilde{j} - 2\tilde{k}$

$\cos\theta = \frac{\tilde{n}_1.\tilde{n}_2}{|\tilde{n}_1||\tilde{n}_2|} = \frac{1}{\sqrt{3}}$, $\sec\theta = \sqrt{3}$

Q18

$$\tilde{a}\times\tilde{b} = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 2 & -4 & 2 \\ 1 & -2 & 3 \end{vmatrix} = -8\tilde{i} - 4\tilde{j}$$

$$A = \frac{1}{2}\left|\tilde{a}\times\tilde{b}\right| = \frac{1}{2}\sqrt{64+16} = 2\sqrt{5}$$

Q19 $\tilde{a} = \overrightarrow{OA} = \tilde{i} + 2\tilde{j} - \tilde{k}$, $\tilde{c} = \overrightarrow{OC} = 3\tilde{i} + m\tilde{j} + \tilde{k}$

$$\tilde{a}\times\tilde{c} = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 1 & 2 & -1 \\ 3 & m & 1 \end{vmatrix} = (2+m)\tilde{i} - 4\tilde{j} + (m-6)\tilde{k}$$

$$A = \left|\tilde{a}\times\tilde{c}\right| = \sqrt{(2+m)^2 + (-4)^2 + (m-6)^2} = 4\sqrt{5}$$

$\therefore\ m^2 - 4m - 12 = 0\ \ \therefore\ m = -2$ or 6

Please inform mathline@itute.com re conceptual and/or mathematical errors.