2023 VCAA Specialist Mathematics Exam 2 Sample Questions Solutions

Section A

Q1 D

Q2 $f=0$, $f=2+2\times 3=8$, $f=2+2\times 8=18$,
$f=2+2\times 18=38$ C

Q3

$$\tilde{a}\times\tilde{b}=\left(\tilde{i}+2\tilde{j}-\tilde{k}\right)\times\left(2\tilde{i}+\tilde{j}-\tilde{k}\right)=\begin{vmatrix}\tilde{i} & \tilde{j} & \tilde{k}\\ 1 & 2 & -1\\ 2 & 1 & -1\end{vmatrix}$$ E

Q4 $\tilde{v}=\overrightarrow{PQ}=8\tilde{i}-5\tilde{j}+14\tilde{k}$

$\tilde{r}(t)=\overrightarrow{OP}+t\tilde{v}=-3\tilde{i}-\tilde{j}-10\tilde{k}+t\left(8\tilde{i}-5\tilde{j}+14\tilde{k}\right)$ A

Q5

$\tilde{n}.(\tilde{r}-\tilde{r}_0)=\left(\tilde{i}-\tilde{j}+3\tilde{k}\right).\left((x-3)\tilde{i}+(y-2)\tilde{j}+(z+4)\tilde{k}\right)=0$

$-x+y-3z=11$ B

Q6 The planes are parallel.

Point $P\left(-\frac{4}{3},0,0\right)$ is on the plane $-15x+12y+36z=20$.

Distance of P from the plane $5x-4y-12z-10=0$ is

$$\frac{\left|5\left(-\frac{4}{3}\right)-10\right|}{\sqrt{5^2+(-4)^2+(-12)_2}}=\frac{50}{3\sqrt{185}}$$ D

Q7 $\mathrm{E}\left(\overline{X}\right)=\mu=18.5$, $\mathrm{sd}\left(\overline{X}\right)=\frac{2}{\sqrt{16}}=0.5$, $c=19.2$

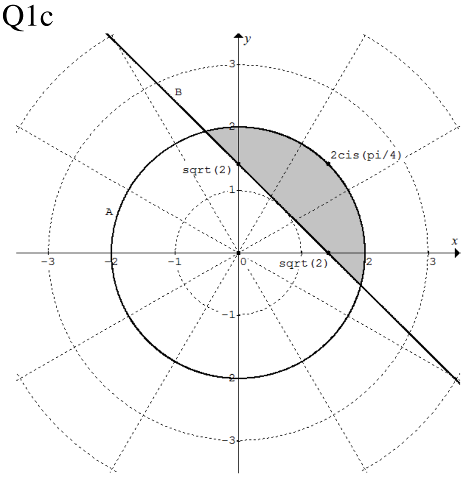
Type II error for the test results when $\bar{x}>19.2$.

$\Pr\left(\overline{X}>19.2\,|\,\mu=18.5\right)\approx 0.08$ A

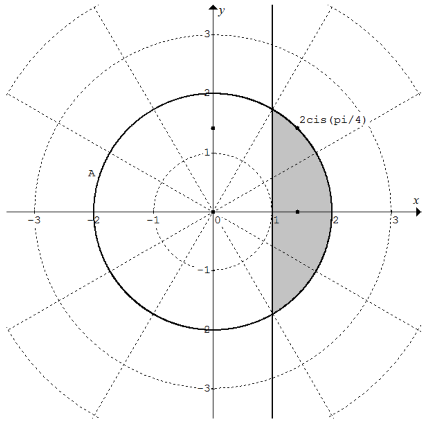
Section B

Q1a Set of complex numbers 'equidistant' from O and $2\text{cis}\left(\frac{\pi}{4}\right)=\sqrt{2}+\sqrt{2}\,i$. In Cartesian form, it is the line $y=-x+\sqrt{2}$

Q1b and Q1c



Q1d



Shaded regions in parts c and d have the same area

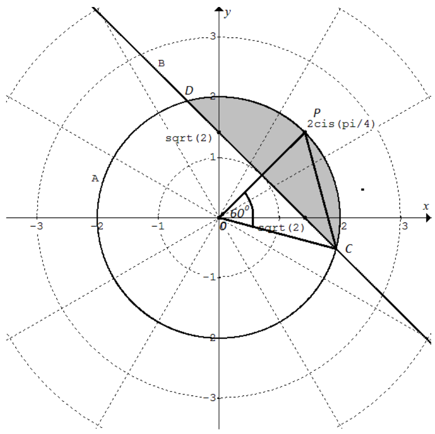
$$=2\times\int_1^2 y\,dx=2\times\int_1^2\sqrt{4-x^2}\,dx\approx 2.4567$$

OR Let $x=2\sin\theta$,

$$=2\times\int_1^2\sqrt{4-x^2}\,dx=2\times\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}4\cos^2\theta\,d\theta$$

$$=4\times\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\cos 2\theta+1)\,d\theta=\frac{4\pi}{3}-\sqrt{3}$$

Alternative Q1d Geometry approach



Let C and D be the intersections of Line B with circle A. Points on Line B (perpendicular bisector of line segment OP) are equidistant from point O and point P $\left(2\text{cis}\frac{\pi}{4}\right)$.

$\therefore$ $CP=CO=OP$ $\therefore$ ΔCOP is equilateral

$\therefore$ $\angle COP=60°$, $\angle COD=120°=\frac{2\pi}{3}$

Shaded segment area = sector area − triangular area

$$=\frac{1}{3}\pi(2)^2-\frac{1}{2}\times 2\times 2\sin\left(\frac{2\pi}{3}\right)=\frac{4\pi}{3}-\sqrt{3}$$

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Q1e Since two of the cube roots are the intersections of the line and the circle, the third one is directly opposite to $2\text{cis}\left(\frac{\pi}{4}\right)$, it is $2\text{cis}\left(-\frac{3\pi}{4}\right)$. All three spaced out at equal argument from each other .: the two intersections (roots) are $2\text{cis}\left(-\frac{\pi}{12}\right)$ and $2\text{cis}\left(\frac{7\pi}{12}\right)$.

Comment on information in Q2:
$\frac{dP}{dt}$ is the growth rate of the population and so is r.
Q2b suggests that r is a *constant* to be evaluated to 2 dec. places.

Q2a $\frac{dP}{dt} = rP\left(1-\frac{P}{30000}\right)$, $r\frac{dt}{dP} = \frac{1}{P} + \frac{1}{30000-P}$,

Given $t=0$, $P=500$.: $rt = \ln\frac{59P}{30000-P}$.:

$P = \frac{30000}{1+59e^{-rt}}$

Q2b When $t=10$, $P=1930$.:

$r = \frac{1}{t}\ln\frac{59P}{30000-P} \approx 0.14$

Q2c When $t=0$,

$\frac{dP}{dt} = rP\left(1-\frac{P}{30000}\right) \approx 0.14(500)\left(1-\frac{500}{30000}\right) \approx 68.9$ /year

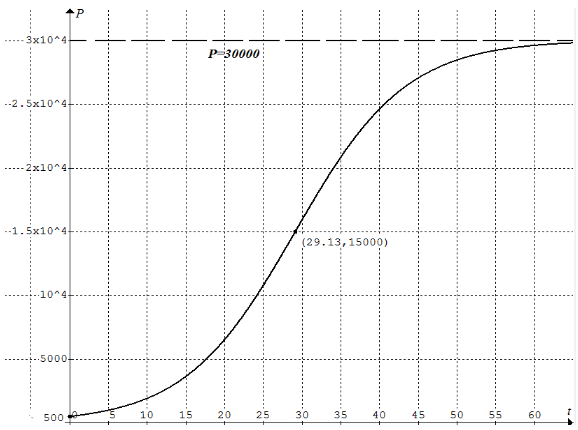
Q2d When $P=10000$,

$t = \frac{1}{r}\ln\frac{59P}{30000-P} \approx \frac{1}{0.14}\ln\frac{59\times 10000}{30000-10000} \approx 24.2$ years

Q2e Fastest population growth, let

$\frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{dP}{dt}\times\frac{d}{dP}\left(\frac{dP}{dt}\right) = \frac{dP}{dt}\times 0.14\left(1-\frac{P}{15000}\right) = 0$

$P = 15000$ and $t \approx 29.13$



Q3a $\tilde{r} = (1+2s+3t)\tilde{i} + (-2-s-2t)\tilde{j} + (2-s+t)\tilde{k}$

$= \left(\tilde{i}-2\tilde{j}+2\tilde{k}\right) + s\left(2\tilde{i}-\tilde{j}-\tilde{k}\right) + t\left(3\tilde{i}-2\tilde{j}+\tilde{k}\right)$

$= \tilde{r}_0 + s\left(2\tilde{i}-\tilde{j}-\tilde{k}\right) + t\left(3\tilde{i}-2\tilde{j}+\tilde{k}\right)$

Q3b Choose two points on Π_1:

Let $s=1$ and $t=0$, $\tilde{r}_1 = 3\tilde{i}-3\tilde{j}+\tilde{k}$

Let $s=0$ and $t=1$, $\tilde{r}_2 = 4\tilde{i}-4\tilde{j}+3\tilde{k}$

$\tilde{a} = \tilde{r}_1 - \tilde{r}_0 = 2\tilde{i}-\tilde{j}-\tilde{k}$, $\tilde{b} = \tilde{r}_2 - \tilde{r}_0 = 3\tilde{i}-2\tilde{j}+\tilde{k}$

$\tilde{n} = \tilde{a}\times\tilde{b} = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 2 & -1 & -1 \\ 3 & -2 & 1 \end{vmatrix} = -3\tilde{i}-5\tilde{j}-\tilde{k}$

Let $\tilde{r} = x\tilde{i}+y\tilde{j}+z\tilde{k}$ and $\tilde{n}.(\tilde{r}-\tilde{r}_0) = 0$

.: Π_1 is $3x+5y+z+5=0$

Q3c Π_2 and Π_1 are parallel .: same $\tilde{n} = -3\tilde{i}-5\tilde{j}-\tilde{k}$

$P(1,0,3)$ is on Π_2 and $\tilde{n}.(\tilde{r}-\tilde{r}_0) = 0$

.: $-3(x-1)-5(y-0)-1(z-3)=0$.: $3x+5y+z-6=0$

Q3di $P(1,0,3)$ is on Π_2 and Π_1 is $3x+5y+z+5=0$

Shortest distance $D = \frac{|3\times 1+5\times 0+1\times 3+5|}{\sqrt{3^2+5^2+1^2}} = \frac{11}{\sqrt{35}}$

Q3dii A unit vector in the direction of $\overrightarrow{QP}$ is

$\frac{3\tilde{i}+5\tilde{j}+\tilde{k}}{\sqrt{3^2+5^2+1^2}} = \frac{1}{\sqrt{35}}\left(3\tilde{i}+5\tilde{j}+\tilde{k}\right)$

$\overrightarrow{OQ} = \overrightarrow{OP} - \overrightarrow{QP} = \left(\tilde{i}+3\tilde{k}\right) - 2\times\frac{11}{\sqrt{35}}\left(\frac{1}{\sqrt{35}}\left(3\tilde{i}+5\tilde{j}+\tilde{k}\right)\right)$

$= -\frac{31}{35}\tilde{i} - \frac{110}{35}\tilde{j} + \frac{83}{35}\tilde{k}$

.: $Q\left(-\frac{31}{35}, -\frac{22}{7}, \frac{83}{35}\right)$

Q4a $\overrightarrow{OP} = \tilde{r}(t=0) = 4\tilde{i}+2\tilde{j}+\tilde{k}$

$\overrightarrow{OQ} = \tilde{r}(s=0) = 5\tilde{i}+4\tilde{j}-2\tilde{k}$

.: $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \tilde{i}+2\tilde{j}-3\tilde{k}$

A unit vector on the line given by

$\tilde{r}(t) = 4\tilde{i}+2\tilde{j}+\tilde{k}+t\left(-\tilde{i}+\tilde{j}+3\tilde{k}\right)$ is $\hat{u} = \frac{-\tilde{i}+\tilde{j}+3\tilde{k}}{\sqrt{11}}$

Vector resolute of $\overrightarrow{PQ}$ in the direction of $\hat{u}$ is

$\left(\overrightarrow{PQ}.\hat{u}\right)\hat{u} = \frac{1}{\sqrt{11}}(-1+2-9)\frac{1}{\sqrt{11}}\left(-\tilde{i}+\tilde{j}+3\tilde{k}\right)$

$= -\frac{8}{11}\left(-\tilde{i}+\tilde{j}+3\tilde{k}\right)$

Vector resolute of $\overrightarrow{PQ} \perp \hat{u} = \overrightarrow{PQ} - \left(-\frac{8}{11}\left(-\tilde{i}+\tilde{j}+3\tilde{k}\right)\right)$

$= \frac{3}{11}\tilde{i} + \frac{30}{11}\tilde{j} - \frac{9}{11}\tilde{k}$

.: shortest

distance $= \sqrt{\left(\frac{3}{11}\right)^2 + \left(\frac{30}{11}\right)^2 + \left(-\frac{9}{11}\right)^2} = \frac{3}{11}\sqrt{110}$

Q4b At the intersection, $\tilde{r}(t) = \tilde{r}(s)$

.: $(1+3t)\tilde{i} + (-3+5t)\tilde{j} + (6-at)\tilde{k}$

$= (-6+4s)\tilde{i} + (2-10s)\tilde{j} + (1+6s)\tilde{k}$

.: Equate components and solve simultaneous equations:

$s = 1,\ t = -1$ and $a = -11$

.: at the intersection, $\tilde{r} = -2\tilde{i} - 8\tilde{j} + 7\tilde{k}$.: $(-2, -8, 7)$

Q4c Normal to the plane $2x - 3y - z = 2$ is

$\tilde{n} = 2\tilde{i} - 3\tilde{j} - \tilde{k}$

It is perpendicular to the line

.: $\left(2\tilde{i} - 3\tilde{j} - \tilde{k}\right).\left(4\tilde{i} + b\tilde{j} + 2\tilde{k}\right) = 0$.; $b = 2$

A point on the line $(1, 1, -5)$, shortest distance to the

plane $= \frac{\left|2\times 1 - 3\times 1 - 1\times(-5) - 2\right|}{\sqrt{2^2 + (-3)^2 + (-1)^2}} = \frac{2}{\sqrt{14}}$

Q5ai $\tilde{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \left(\tilde{i} + 3\tilde{j} - 2\tilde{k}\right) \times \left(2\tilde{j} - \tilde{k}\right) = \tilde{i} + \tilde{j} + 2\tilde{k}$

Q5aii Let $R(x, y, z)$ be a point on the plane Π_1.

$\tilde{n}.\overrightarrow{AR} = \left(\tilde{i} + \tilde{j} + 2\tilde{k}\right).\left((x-1)\tilde{i} + (y-0)\tilde{j} + (z-2)\tilde{k}\right) = 0$

.: $x + y + 2z = 5$

Q5bi Solve $x + y + 2z = 5$ and $x - y - z = 0$, $x = 0$ at *y-z* plane. .: $y = -5$, $z = 5$.: $P(0, -5, 5)$

Q5bii Normal to Π_1 $x + y + 2z = 5$ is $\tilde{n}_1 = \tilde{i} + \tilde{j} + 2\tilde{k}$

Normal to Π_2 $x - y - z = 0$ is $\tilde{n}_2 = \tilde{i} - \tilde{j} - \tilde{k}$

Vector parallel to L is

$$\tilde{v} = \tilde{n}_1 \times \tilde{n}_2 = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 1 & 1 & 2 \\ 1 & -1 & -1 \end{vmatrix} = \tilde{i} + 3\tilde{j} - 2\tilde{k}$$

Vector equation of L is $\tilde{r} = \overrightarrow{OP} + t\left(\tilde{i} + 3\tilde{j} - 2\tilde{k}\right)$

.: $\tilde{r} = \left(-5\tilde{j} + 5\tilde{k}\right) + t\left(\tilde{i} + 3\tilde{j} - 2\tilde{k}\right)$

Q5biii Point $A(1, 0, 2)$, plane Π_2 $x - y - z = 0$

$D = \frac{\left|1\times 1 + 0(-1) + 2(-1)\right|}{\sqrt{1^2 + (-1)^2 + (-1)^2}} = \frac{1}{\sqrt{3}}$

Q5biv $\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = \tilde{i} + 5\tilde{j} - 3\tilde{k}$

Vector resolute of $\overrightarrow{PA}$ in the direction of $\tilde{v}$ is

$\left(\overrightarrow{PA}.\hat{v}\right)\hat{v} = \frac{11}{7}\left(\tilde{i} + 3\tilde{j} - 2\tilde{k}\right)$

Vector resolute of $\overrightarrow{PA}$ perpendicular to $\tilde{v}$ is

$\overrightarrow{PA} - \left(\overrightarrow{PA}.\hat{v}\right)\hat{v} = \frac{11}{7}\left(\tilde{i} + 3\tilde{j} - 2\tilde{k}\right) = -\frac{4}{7}i + \frac{2}{7}j + \frac{1}{7}k$

$D = \frac{1}{7}\sqrt{(-4)^2 + 2^2 + 1^2} = \frac{\sqrt{21}}{7}$

Q6a Let $4\sqrt{2}\sin\left(\frac{\pi t}{2}\right) + 4\sqrt{2} = 0$. $t = 3$

Q6b $\tilde{r}_S(3) = 69\tilde{i} + 15\tilde{j}$, $\left|\tilde{r}_S(3)\right| \approx 70.6\,\text{cm}$

Q6c $\tilde{v}_S = \frac{d\tilde{r}_S}{dt} = 23\tilde{i} + 5\tilde{j} + 2\sqrt{2}\pi\cos\left(\frac{\pi t}{2}\right)\tilde{k}$

$|\tilde{v}| = \sqrt{23^2 + 5^2 + \left(2\sqrt{2}\pi\right)^2\cos^2\left(\frac{\pi t}{2}\right)}$

When $\cos^2\left(\frac{\pi t}{2}\right) = 1$,

$|\tilde{v}|_{\max} = \sqrt{23^2 + 5^2 + \left(2\sqrt{2}\pi\right)^2} \approx 25.2\ \text{cm s}^{-1}$

Q6d $\tilde{r}_M(t) = \int\left(6\tilde{i} + \tilde{j} + \frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right)\tilde{k}\right)dt$

$\tilde{r}_M(t) = 6t\,\tilde{i} + t\,\tilde{j} + \sin\left(\frac{\pi t}{6}\right)\tilde{k} + 10\tilde{i} + 4\tilde{j} + 4\sqrt{2}\tilde{k}$

Q6e $\tilde{r}_S(t_S) = 23t_S\,\tilde{i} + 5t_S\,\tilde{j} + \left(4\sqrt{2}\sin\left(\frac{\pi t_S}{2}\right) + 4\sqrt{2}\right)\tilde{k}$

$\tilde{r}_M(t_M) = 6t_M\,\tilde{i} + t_M\,\tilde{j} + \sin\left(\frac{\pi t_M}{6}\right)\tilde{k} + 10\tilde{i} + 4\tilde{j} + 4\sqrt{2}\tilde{k}$

Same position: $\tilde{r}_S(t_S) = \tilde{r}_M(t_M)$

.: $23t_S = 6t_M + 10$ and $5t_S = t_M + 4$

.: $t_S = 2$ and $t_M = 6$

At $t_S = 2$, $\tilde{r}_S = 46\tilde{i} + 10\tilde{j} + 4\sqrt{2}\tilde{k}$

At $t_M = 6$, $\tilde{r}_M = 46\tilde{i} + 10\tilde{j} + 4\sqrt{2}\tilde{k}$

.: same position $\left(46, 10, 4\sqrt{2}\right)$

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