2023 VCAA Specialist Mathematics Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
C	B	E	B	E	C	D	A	D	A

11	12	13	14	15	16	17	18	19	20
E	A	E	B	D	D	C	C	B	A

Q1 C

Q2 Asymptote $x=1$ indicates that $a+b+c=0$.
Asymptote $y=2x+1$ indicates that $a=\frac{1}{2}$ and b is a negative value. B

Q3 Sketch the graph of $y=\sec(x)$ in the interval $-\pi \le x \le \pi$.
Translate the graph vertically to find a. E

Q4 Let $w=-2a+2ai$.: $z=w-1$.: $\bar{z}=\bar{w}-1$
.: $1+\bar{z}=\bar{w}=-2a-2ai$.: $\frac{4a}{-2a-2ai}=\frac{2}{-1-i}=\sqrt{2}\text{cis}\frac{3\pi}{4}$ B

Q5 z is in the first quadrant.
$\arg(z^3)=-\pi$.: $\arg(z^3)=\pi$.: $\arg(z)=\frac{\pi}{3}$.: $z=4\text{cis}\left(\frac{\pi}{3}\right)$
$z^2=16\text{cis}\left(\frac{2\pi}{3}\right)=-4\times 4\text{cis}\left(-\frac{\pi}{3}\right)=-4\bar{z}$ E

Q6 First: 0.5, second: 1.1420, third: 2.709 C

Q7 Sketch a smooth curve through $(-1,2)$ without crossing the tangent line segments.
The curve passes through $(1.5,1.0)$ approximately. D

Q8 Q at time t in the pool of volume $8000-5t$
.: concentration $=\frac{Q}{8000-5t}$.: $\frac{dQ}{dt}=-\frac{20Q}{8000-5t}=\frac{4Q}{t-1600}$ A

Q9 At $t=2$, $\frac{dx}{dt}=\frac{2}{3}$ and $\frac{dy}{dt}=\frac{1}{2}$.: $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3}{4}$ D

Q10 Integration by parts: $I_n=nI_{n-1}+\left[(1-x)^n e^x\right]_0^1$
.: $I_n=nI_{n-1}-1$ A

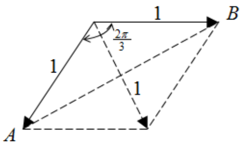
Q11 $2\pi\int_0^{\frac{\pi}{2}}\cos y\sqrt{1+\sin^2 y}\,dy=2\pi\int_0^1\sqrt{1+u^2}\,du$ E

Q12 $\frac{dv}{dt}=1+v$, $v=0$ when $t=0$, $t=\int\frac{1}{1+v}\,dv$.: $t=\log_e(1+v)$
When $t=\log_e(e+1)$, $v=e$ A

Q13 $u=2.5$, $s=-80$, $a=-9.8$ Use $s=ut+\frac{1}{2}at^2$ to find $t\approx 4.30$ E

Q14 $\tilde{n}$ is a unit vector perpendicular to both $\tilde{a}$ amd $\tilde{b}$.
Since both are on the $\tilde{i}$ - $\tilde{j}$ plane .: $\tilde{n}=\tilde{k}$.: $|\tilde{c}.\tilde{n}|=3$ B

Q15 $\left|\overrightarrow{AB}\right|=2\sin\frac{\pi}{3}=\sqrt{3}$ D



Q16 Upward component: $z=15t-4.9t^2+1.5$
$v_z=\frac{dz}{dt}=15-9.8t$ At the highest point $\frac{dz}{dt}=0$, $t=\frac{15}{9.8}$ and $z\approx 12.98$. At $t=0$, $z=1.5$
.: total vertical distance $\approx 2\times 12.98-1.5=24.46$ D

Q17 $2\tilde{i}-7\tilde{j}+\gamma\tilde{k}=\begin{vmatrix}\tilde{i} & \tilde{j} & \tilde{k}\\ \alpha & 1 & -1\\ 3 & \beta & 4\end{vmatrix}=(4+\beta)\tilde{i}-(4\alpha+3)\tilde{j}+(\alpha\beta-3)\tilde{k}$
Equate components C

Q18 $\tilde{n}_1=2\tilde{i}-k\tilde{j}+3\tilde{k}$, $\tilde{n}_2=2k\tilde{i}+3\tilde{j}-2\tilde{k}$
Let $\tilde{n}_1.\tilde{n}_2=0$.: $k=6$ C

Q19 Sample $n=16$, $\text{E}(\bar{X})=\mu=800$, $\text{sd}(\bar{X})=\frac{200}{\sqrt{16}}=50$
$\bar{x}=\frac{13500}{16}=843.75$, $\Pr(\bar{X}>843.75)\approx 0.1908$ B

Q20 $\text{sd}(\bar{X})=\frac{\sigma}{\sqrt{100}}=\frac{\sigma}{10}$, $\bar{x}=\frac{10500+15500}{2}=13000$
For 99% confidence interval, $\Pr(Z<z)=0.995$.: $z\approx 2.57583$
.: $13000+2.57583\times\frac{\sigma}{10}\approx 15500$.: $\sigma\approx 9710$ A

SECTION B

Q1a At $C(1,0)$, $-1(1+a)^2=0$.: $a=-1$
As $x\to 1$, $x-e^{x-1}\to 0$ and when $b=0$, $e^{x-1}-x+b\to 0$
.: the curves meet at $C(1,0)$.

Q1b $\frac{d}{dx}\left(-x(x-1)^2\right)=-(x-1)(3x-1)=0$ at $x=1$
$\frac{d}{dx}\left(e^{x-1}-x\right)=e^{x-1}-1$ As $x\to 1$, $e^{x-1}-1\to 0$
.: the curves meet smoothly at $x=1$

Q1ci $\frac{d}{dx}\left(-x(x-1)^2\right)=-(x-1)(3x-1)=0$ at $x=\frac{1}{3}$ also, $y=-\frac{4}{27}$
i.e. at $A\left(\frac{1}{3},-\frac{4}{27}\right)$

Q1cii Let second derivative be zero. $-6x+4=0$.: $x=\frac{2}{3}$ and

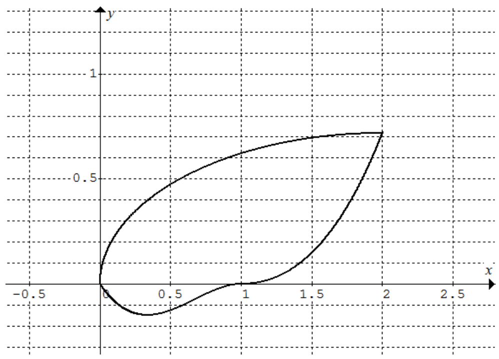
$y=-\frac{2}{27}$.: $B\left(\frac{2}{3},-\frac{2}{27}\right)$

Q1d $\frac{x-2}{2}=\cos t$, $\frac{y}{e-2}=\sin t$, where $\frac{\pi}{2}\le t\le\pi$

.: $\left(\frac{x-2}{2}\right)^2+\left(\frac{y}{e-2}\right)^2=1$ centred at $(2,0)$

When $t=\frac{\pi}{2}$, $x=2$ and $y=e-2$; when $t=\pi$, $x=0$ and $y=0$

Q1e



Q1fi $\int_{\frac{\pi}{2}}^{\pi}\sqrt{(-2\sin t)^2+((e-2)\cos t)^2}\;dt$

Q1fii 2.255 km

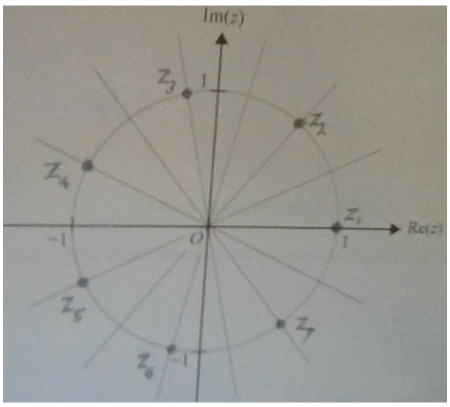
Q2a $w^7-1=\left(\text{cis}\frac{2\pi}{7}\right)^7-1=\text{cis}\frac{7(2\pi)}{7}-1=1-1=0$

.: w is a root of $z^7-1=0$

Q2b $\text{cis}\frac{4\pi}{7}$, $\text{cis}\frac{6\pi}{7}$, $\text{cis}\frac{8\pi}{7}$, $\text{cis}\frac{10\pi}{7}$, $\text{cis}\frac{12\pi}{7}$, $\text{cis}\frac{14\pi}{7}$

or $\text{cis}\frac{4\pi}{7}$, $\text{cis}\frac{6\pi}{7}$, $\text{cis}\frac{-6\pi}{7}$, $\text{cis}\frac{-4\pi}{7}$, $\text{cis}\frac{-2\pi}{7}$, $\text{cis}0$

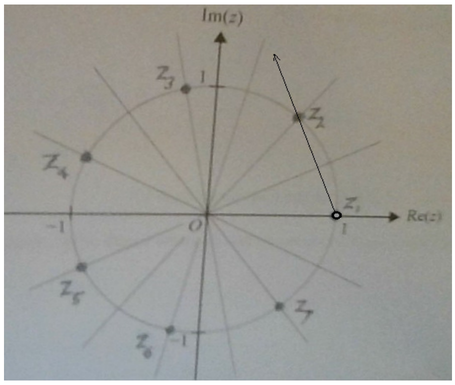
Q2c



where $z_1=\text{cis}0$, $z_2=\text{cis}\frac{2\pi}{7}$ $z_3=\text{cis}\frac{4\pi}{7}$, $z_4=\text{cis}\frac{6\pi}{7}$,

$z_5=\text{cis}\frac{-6\pi}{7}$, $z_6=\text{cis}\frac{-4\pi}{7}$, $z_7=\text{cis}\frac{-2\pi}{7}$

Q2di



Q2dii $\frac{2\pi}{7}+\phi=\pi-\phi=\theta$, $\phi=\frac{5\pi}{14}$.: $\theta=\pi-\frac{5\pi}{14}=\frac{9\pi}{14}$

Ray: $\text{Arg}(z-1)=\frac{9\pi}{14}$

Q2e Verify by expansion:

$(z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$

$=z(z^6+z^5+z^4+z^3+z^2+z+1)-(z^6+z^5+z^4+z^3+z^2+z+1)$

$=(z^7+z^6+z^5+z^4+z^3+z^2+z)-(z^6+z^5+z^4+z^3+z^2+z+1)$

$=z^7-1=0$

Q2fi $\text{cis}\frac{2\pi}{7}+\text{cis}\frac{12\pi}{7}=\text{cis}\frac{2\pi}{7}+\text{cis}\frac{-2\pi}{7}=2\cos\frac{2}{7}\pi$

Q2fii From part e: $z^6+z^5+z^4+z^3+z^2+z+1=0$

$(z^6+z)+(z^4+z^3)+(z^5+z^2)=-1$

.: $2\cos\frac{2\pi}{7}+2\cos\frac{4\pi}{7}+2\cos\frac{6\pi}{7}=-1$

.: $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$

Q3ai $\int_2^5\pi y^2\,dx=\int_2^5\pi(x-1)\,dx$

Q3aii $V=\frac{15\pi}{2}$

Q3bi $\frac{dy}{dx}=\frac{1}{2\sqrt{x-1}}$ Surface area $=\int_2^5 2\pi\sqrt{x-1}\sqrt{1+\left(\frac{1}{2\sqrt{x-1}}\right)^2}\;dx$

$=\pi\int_2^5\sqrt{4(x-1)}\sqrt{1+\frac{1}{4(x-1)}}\;dx=\pi\int_2^5\sqrt{4x-3}\;dx$

Q3bii Surface area ≈ 30.846

Q3c Efficiency ratio $\approx\frac{\pi 1^2+\pi 2^2+30.846}{\frac{15\pi}{2}}\approx 1.98$

Q3d $V=\int_2^k\pi(x-1)\,dx=24\pi$, $k=8$

Surface area $=\pi\int_2^8\sqrt{4x-3}\;dx\approx 24.1646\pi$

Efficiency ratio $\approx\frac{\pi 1^2+\pi\left(\sqrt{7}\right)^2+24.1646\pi}{24\pi}\approx 1.34$

Q4a $\int \frac{1}{P\left(1-\frac{P}{1000}\right)} dP = \int dt$, $\frac{A}{P} + \frac{B}{1-\frac{P}{1000}} = \frac{1}{P\left(1-\frac{P}{1000}\right)}$

$\frac{A\left(1-\frac{P}{1000}\right)+BP}{P\left(1-\frac{P}{1000}\right)} = \frac{1}{P\left(1-\frac{P}{1000}\right)}$ $\therefore A = 1$ and $B = \frac{1}{1000}$

Q4b $t=0$, $P=200$, $P = \frac{1000}{1+4e^{-t}}$ $\therefore D = 4$

Q4c $t=0$, $Q=n$, $Q = \frac{1000}{1+9e^{-1.1t}}$ $\therefore n = \frac{1000}{1+9} = 100$

Q4d $t=6$, $Q = \frac{1000}{1+9e^{-1.1\times 6}} \approx 988$

Q4ei $\frac{dQ}{dt} = \frac{11}{10} Q\left(1-\frac{Q}{1000}\right)$, $\frac{d^2Q}{dt^2} = \frac{11}{10}\left(1-\frac{Q}{500}\right)\frac{dQ}{dt}$

Q4eii For max $\frac{dQ}{dt}$, let $\frac{d^2Q}{dt^2} = \frac{11}{10}\left(1-\frac{Q}{500}\right)\frac{dQ}{dt} = 0$ $\therefore Q = 500$

$Q = \frac{1000}{1+9e^{-1.1t}} = 500$ $\therefore t \approx 2$

Q4f

Q4g Let $\frac{dQ}{dt} = \frac{11}{10} Q\left(1-\frac{Q}{1000}\right) - 0.055Q = 0$, $Q = 950$

Q5a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \tilde{j} + \tilde{k}$, $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\tilde{i} + \hat{j} + 2\tilde{k}$

$\text{Area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|\left(\begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{vmatrix}\right)\right| = \frac{1}{2}\left|\tilde{i} + 2\tilde{j} - 2\tilde{k}\right| = \frac{3}{2}$

Q5b Let $\tilde{u}$ be a unit vector in the direction of $\overrightarrow{AC}$,

$\tilde{u} = \frac{1}{3}\left(2\tilde{i} + \hat{j} + 2\tilde{k}\right)$. $\overrightarrow{AB}.\tilde{u} = 1$ and $AB = \sqrt{2}$

$\therefore$ shortest distance $= \sqrt{\left(\sqrt{2}\right)^2 - 1^2} = 1$

Q5c Vector perpendicular to ψ is $\tilde{n} = 2\tilde{i} - 2\tilde{j} - \tilde{k}$

The given line is parallel to $\tilde{b} = \tilde{i} - 2\tilde{j} + 2\tilde{k}$.

Angle ϕ between $\tilde{b}$ and $\tilde{n}$: $\tilde{n}.\tilde{b} = |\tilde{n}||\tilde{b}|\cos\phi$, $\phi \approx 63.61°$

Angle θ between $\tilde{b}$ and plane ψ: $\theta = 90° - 63.61° \approx 26°$

Q5d Vector equation of L is $\tilde{r} = t\,\tilde{n} = 2t\,\tilde{i} - 2t\,\tilde{j} - t\,\tilde{k}$

Parametric form: $x = 2t$, $y = -2t$, $z = -t$

Q5e The point D is on L and ψ $\therefore 2(2t) - 2(-2t) - (-t) = -18$

$\therefore t = -2$ $\therefore \overrightarrow{OD} = -4\tilde{i} + 4\tilde{j} + 2\tilde{k}$

Shortest distance from O to $\psi = \left|\overrightarrow{OD}\right| = 6$

Q5f $\overrightarrow{OD} = -4\tilde{i} + 4\tilde{j} + 2\tilde{k}$ $\therefore D(-4, 4, 2)$

Q6a Population $\sigma = 1$

Sample $n = 20$, $\bar{x} = 11.39$, $\text{sd}\left(\overline{X}\right) = \frac{1}{\sqrt{20}}$

95% confidence interval $\left(11.39 - 1.96 \times \frac{1}{\sqrt{20}},\ 11.39 + 1.96 \times \frac{1}{\sqrt{20}}\right)$

simplify to $(10.95, 11.83)$

Q6b 95% of $60 = 57$

Q6c Required width $=$ 40% of $= 2 \times 1.96 \times \frac{1}{\sqrt{20}}$

$\therefore 2 \times 1.96 \times \frac{1}{\sqrt{n}} = 0.40 \times 2 \times 1.96 \times \frac{1}{\sqrt{20}}$ $\therefore n = 125$

Q6d H_0: $\mu = 12$, H_1: $\mu < 12$

Q6ei $\text{E}\left(\overline{X}\right) = \mu = 12$, $\text{sd}\left(\overline{X}\right) = \frac{1}{\sqrt{40}}$, $\bar{x} = 11.6$

p-value $= \Pr\left(\overline{X} \le 11.6 \mid \mu = 12\right) \approx 0.0057$

Q6eii Since the p-value is less than 0.01

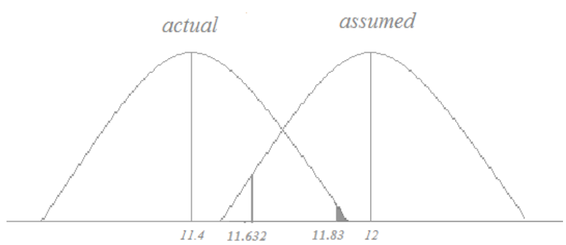
$\therefore$ strong evidence against H_0

Q6f p-value $= \Pr\left(\overline{X} \le c \mid \mu = 12\right) \ge 0.01$, $c \ge 11.63217$

Critical sample mean $\bar{x} \approx 11.632$

Q6g $\Pr\left(\overline{X} \ge 11.63217 \mid \mu = 11.4\right) \approx 0.071$

Q6h Type II error: H_1 is considered as H_0



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