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2023 Mathematical Methods

Year 12

Application Task

Time allowed: 4 hours plus

Application Task

Theme: Quartic functions as products of quadratic functions

Assumed knowledge:

Algebra, polynomial functions, quartic functions, quadratic functions and discriminants, graphs, binomial expansion, factorization, transformations, co-ordinates, parameters, calculus, CAS

Introduction:

The task is an investigation of polynomials $P(x)$ of a single real variable x.

Only $P(x)=(x^2+px+q)(x^2+rx+s)$, i.e. the products of two irreducible quadratic functions, are considered for parameters p, q, r and $s \in R$.

The features/behaviour of $P(x)$ are explored/analysed when parameters p, q, r and s are varied.

Part I (80 minutes plus)

In Part I, consider $P(x)=(x^2+px+q)(x^2+rx+s)$ under the following constraints.

Constraint A: $p^2-4q>0$, $r^2-4s>0$, $p \neq r$ and $q \neq s$

Constraint B: $p^2-4q<0$, $r^2-4s<0$, $p \neq r$ and $q \neq s$

a. Given $P(x)=x^4+3x^3-21x^2-43x+60=(x^2+2x-3)Q(x)$, find $Q(x)$.
Hence express $P(x)$ as the products of two quadratic functions in **three other** ways.

b. Sketch the graph of $P(x)=x^4+3x^3-21x^2-43x+60$ showing its important features.
Correct coordinates to 2 decimal places.

c. Given $f(x)=(x^2-5x+4)(x^2+bx+c)$ satisfies Constraint A, i.e. $b^2-4c>0$, $b\neq -5$ and $c\neq 4$, investigate the effects on the graph of $f(x)$ when b varies and c stays constant. Graph sketching is not required.

d. Find the values of b and c such that the graph of $f(x)=(x^2-5x+4)(x^2+bx+c)$ has two positive and two negative x-intercepts.

e. Find the values of b and c such that the graph of $f(x)=(x^2-5x+4)(x^2+bx+c)$ has two x-intercepts only when Constraint A is removed. Note: There are two possibilities.

f. Given $g(x)=(x^2-4x+5)(x^2+vx+w)$ satisfies Constraint B, i.e. $v^2-4w<0$, $v\neq -4$ and $w\neq 5$, investigate the effects on the graph of $g(x)$ when v varies and w stays constant. Graph sketching is in part **g**.

g. Select an appropriate value for w and three values for v, sketch the graphs of $g(x)=(x^2-4x+5)(x^2+vx+w)$ to display your findings in part f.
Clearly label each graph with its v value, and coordinates of feature points (correct to 2 decimal places).

h. Identify/outline/discuss two important differences between the graphs of $f(x)$ and the graphs of $g(x)$

End of Part I

Part II (80 minutes plus)

In Part I, you investigated $P(x) = (x^2 + px + q)(x^2 + rx + s)$ under the following constraints.

Constraint A: $p^2 - 4q > 0$, $r^2 - 4s > 0$, $p \neq r$ and $q \neq s$

Constraint B: $p^2 - 4q < 0$, $r^2 - 4s < 0$, $p \neq r$ and $q \neq s$

Now, you are to investigate $P(x) = (x^2 + px + q)(x^2 + rx + s)$ under the following constraints.

Constraint C: $p^2 - 4q = 0$, $r^2 - 4s = 0$, $p \neq r$ and $q \neq s$

Constraint D: $p^2 - 4q = 0$, $r^2 - 4s = 0$, $p = r$ and $q = s$

a. Select appropriate values of p, q, r and s such that $P(x) = (x^2 + px + q)(x^2 + rx + s)$ satisfies Constraint C.
Sketch the graph of $P(x)$ for your selected values of p, q, r and s.
Label important points with coordinates.

b. Let $P(x) = (x^2 + px + q)(x^2 + rx + s)$ satisfies Constraint C and $p > r$.
Write down the relationship between p and r such that the x-intercepts are α units apart.
Hence, show that $\sqrt{q} - \sqrt{s} = \alpha$.

c. Discuss the effects of increasing the value of α on the local maximum of $P(x)$ in part **a**.
Illustrate your answer by sketching two more graphs for different values of α on the same axes in part **a**.

d. Describe the change in the graph of $P(x)=(x^2+px+q)(x^2+rx+s)$ satisfying Constraint C as $\alpha \to 0$.

Now consider $P(x)=(x^2+px+q)(x^2+rx+s)$ under Constraint D.

e. Express $P(x)$ in terms of p only.

Write down the coordinates of the minimum point in terms of p.

f. Give an example of $P(x)=(x^2+px+q)(x^2+rx+s)$ under Constraint D.

Sketch its graph and show the coordinates of the axis-intercepts.

g. $U(x)=x^4+\beta x^3+\gamma x^2+\delta x+\varepsilon=P(x)+\sigma$ where $P(x)$ satisfies Constraint D, and $\beta, \gamma, \delta, \varepsilon$ and σ are real coefficients.

Express β, γ, δ and ε in terms of p and/or σ.

h. $U(x)$ is the image of transformed x^4. What are the transformations?

i. Consider $h(x)=x^4-3x^3+\frac{15}{4}x^2-\frac{15}{8}x+7$ and $k(x)=x^4-3x^3+\frac{27}{8}x^2-\frac{27}{16}x+7$.
Determine which one of $h(x)$ and $k(x)$ satisfy Constraint D, and express it in $(x+\mu)^4+\lambda$ form where μ and $\lambda \in R$.

End of Part II

Part III (80 minutes plus)

In Part I and Part II, you investigated $P(x)=(x^2+px+q)(x^2+rx+s)$ under

Constraint A: $p^2-4q>0$, $r^2-4s>0$, $p\neq r$ and $q\neq s$

Constraint B: $p^2-4q<0$, $r^2-4s<0$, $p\neq r$ and $q\neq s$

Constraint C: $p^2-4q=0$, $r^2-4s=0$, $p\neq r$ and $q\neq s$

Constraint D: $p^2-4q=0$, $r^2-4s=0$, $p=r$ and $q=s$

In Part III, you are to investigate

(i) $P(x)=(x^2+px+q)(x^2+rx+s)$ under Constraint E: $p^2-4q<0$, $r^2-4s>0$

(ii) biquadratic functions $B(x)=x^4+bx^2+c$ where $b, c\in R$.

Two examples are $F(x)=x^4-3x^2+2$ and $G(x)=x^4-3x^2+9$

a. Let $p=q=s=2$ such that $P(x)=(x^2+2x+2)(x^2+rx+2)$ satisfies Constraint E. Find the possible values of r.

b. Explore the behaviour of $P(x)$ in part **a** when r is varied systematically by sketching graphs of $P(x)$ for several appropriately selected values of r.
Comment on the axis-intercepts, stationary points and points of inflection if they exist.

c. At a point of inflection, the second derivative of $P(x)$ in part **a**, $P''(x)=0$.
Find the values of r for point(s) of inflection to exist.

d. Find the x-coordinates of the points of inflection of $P(x)$ in part **a** when $r=4$.

Now consider biquadratic functions, $B(x)=x^4+bx^2+c$ where $b, c \in R$.
Two examples are $F(x)=x^4-3x^2+2$ and $G(x)=x^4-3x^2+9$

e. Express each of $F(x)$ and $G(x)$ as product of two quadratic functions.

f. Sketch the graphs of $F(x)$ and $G(x)$ on the same set of axes.

g. Discuss the possible number of x-axis intercepts for biquadratic functions, $B(x) = x^4 + bx^2 + c$ where $b, c \in R$.
State the value(s) of $b^2 - 4c$ for each number of x-axis intercepts.

h. Discuss whether **all** quartic functions can be expressed as product of two quadratic functions.

End of Part III
End of Application Task