itute.com

online & home tutors Registered business name: itute ABN: 96 297 924 083

2024

Mathematical Methods

Trial Examination 2
(2 hours)

SECTION A Multiple-choice questions

Instructions for Section A

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 In the interval $[0,1]$, $f(x)=12x^3-14x^2-1$ has

A. has one x-intercept and two stationary points.

B. has no x-intercept, two stationary points and one inflection point.

C. has no x-intercept, two stationary points and no inflection point.

D. has one y-intercept, one stationary point and one inflection point.

E. has one x-intercept, one y-intercept, one stationary point and one inflection point.

Question 2 The average gradient of the strictly increasing section of the graph of $y=\frac{\pi}{2}\sin\left(\frac{2x}{5}-\frac{\pi}{4}\right)-1$ is

A. $\frac{2}{5}$

B. $\frac{\pi}{6}$

C. $\frac{3}{2\pi}$

D. $\frac{4}{3\pi}$

E. $\frac{5}{4\pi}$

Question 3 $y=\pm\sqrt{1-x^2}$ is the equation of a unit circle centred at $(0,0)$. The area bounded by $y=\frac{1}{\sqrt{2}}$ and $y=\sqrt{1-x^2}$ is

A. $\frac{\pi^2}{4}$

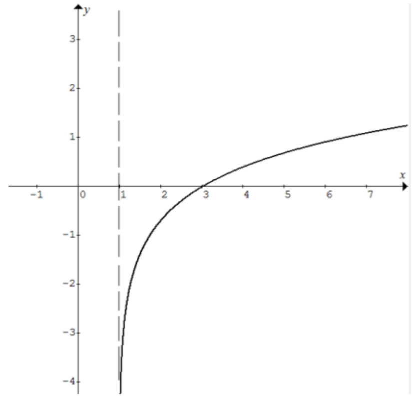
B. $\frac{\pi+2}{4}$

C. $\frac{\pi+1}{2}$

D. $\frac{\pi-1}{4}$

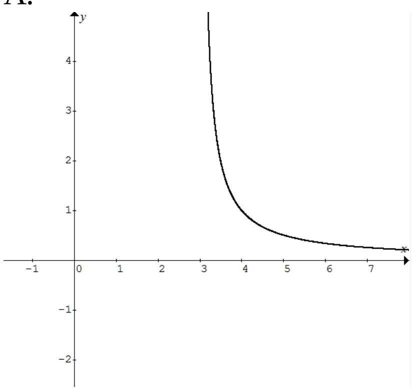
E. $\frac{\pi-2}{4}$

Question 4

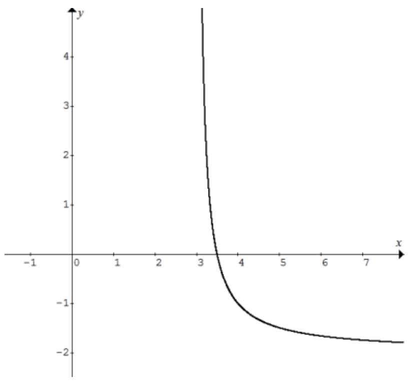


The derivative of the graph above is best represented by Graph

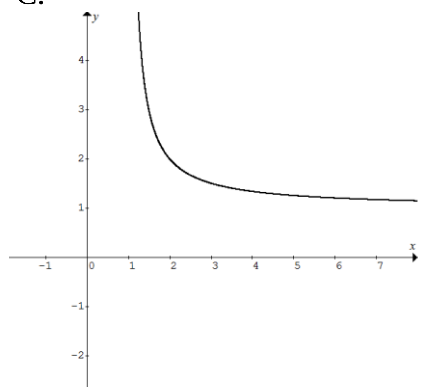
A.



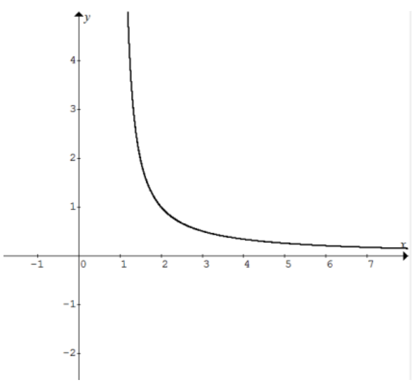
B.



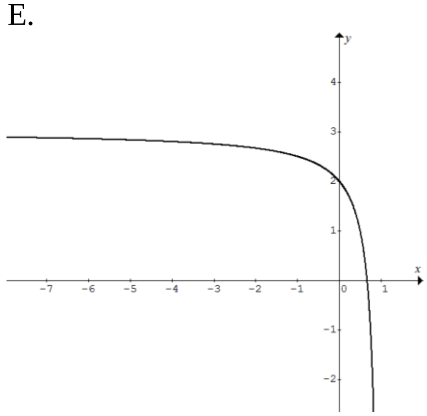
C.



D.



E.



Question 5 The inverse of $\log_{10}\left(1+10^{\sqrt{x}}\right)$ for $x \in (0,1)$ is

A. $\left(\log_{10}\left(10^x-1\right)\right)^2,\ x \in (0,1)$

B. $\left(\log_{10}\left(10^x-1\right)\right)^2,\ x \in (\log_{10}2, \log_{10}11)$

C. $\log_{10}\left(10^x-1\right)^2,\ x \in (0,1)$

D. $\log_{10}\left(10^x-1\right)^2,\ x \in (\log_{10}2, \log_{10}11)$

E. $2\log_{10}\left(10^x-1\right),\ x \in (2,11)$

Question 6 Equation $a^2 \sin^2(x) - a\sin(x) = 2$ has exactly two solutions for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ if

A. $-2 < a \le 1$

B. $-2 \le a \le 2$

C. $a \le -2$ or $a \ge 2$

D. $-1 \le a \le 1$

E. $a < -1$ or $a > 1$

Question 7 The graph of $f(x) = x^3 + 3x + c$ is translated 2 units in the negative x direction, then dilated in the x direction by a factor of 2, and then reflected in the line $y = x$. After the sequence of transformations, the coordinates of the inflection point of $f(x)$ becomes

A. $(-3, -4)$

B. $(-4, -3)$

C. $(-3, c)$

D. $(c, -4)$

E. $(-3, -c)$

Question 8 The graphs of $y = e^{kx}$ and $ky = \log_e x$ have exactly one intersection

A. when $0 < k < e$

B. when $k \le 0$ or $k \ge e^{-1}$

C. when $k < e^{-1}$

D. when $k \le 0$ or $k = e^{-1}$

E. when $k \le e^{-1}$

Question 9 The two curves in $y = \begin{cases} 0.5(x-1)^2 + 1.5, & x < 2 \\ a(x-4)^2 + b, & x \ge 2 \end{cases}$ are joined smoothly at $x = 2$.

The value of b is

A. $b = 3$

B. $b = 2$

C. $b = 1$

D. $b = 0$

E. $b = -1$

Question 10 Given that line $y = 2 - \sqrt{3}x$ and line $y = 3 - \frac{1}{\sqrt{3}}x$ intersect at point P.

Let L be a line which bisects the angle between the two given lines at P. A possible gradient of line L is

A. $\sqrt{2}$

B. $\frac{3}{2}$

C. $\frac{1}{\sqrt{2}}$

D. $-\sqrt{2}$

E. -1

Question 11 For $b > a > 0$ and $n \in Q^+$, $\int_a^b x^n \, dx = F(b) - F(a)$ represents the area of a region enclosed by the x-axis, the curve $y = x^n$ and the lines $x = a$ and $x = b$. The region is dilated in the x-direction by factor h and in the y-direction by factor k. The area of the transformed region is given by

A. $\frac{k}{h^n}\left(F(hb) - F(ha)\right)$

B. $\frac{k}{h}\left(F(hb) - F(ha)\right)$

C. $\frac{k}{h^n}\left(F(kb) - F(ka)\right)$

D. $\frac{k}{h}\left(F(b) - F(a)\right)$

E. $hk\left(F(b) - F(a)\right)$

Question 12 Let $h(x) = f(x) \times g(x)$ where $f(x) = 1 - \frac{\sin(x)}{\cos(x)}$ and $g(x) = 1 + \frac{\cos(x)}{\sin(x)}$.

$h'(x) =$

A. $\frac{1}{\cos^2(x)} - \frac{1}{\sin^2(x)}$

B. $\frac{1}{\sin^2(x)} - \frac{1}{\cos^2(x)}$

C. $-\frac{1}{\sin^2(x)} - \frac{1}{\cos^2(x)}$

D. $\frac{1}{\sin^2(x)} + \frac{1}{\cos^2(x)}$

E. $\frac{1}{\sin(x)} - \frac{1}{\cos(x)}$

Question 13 $f(x)=\sin\left(\frac{x}{m}\right)+\cos\left(\frac{x}{n}\right)$ is a periodic function, where m and n are different prime numbers.

The period of $f(x)$ is

A. 2π

B. $mn\pi$

C. $2(m-n)\pi$

D. $(m+n)\pi$

E. $2mn\pi$

Question 14 Let $f(x)=x(x^2-x-2)^{-1}$, $g(x)=\log_e\left(\frac{1}{x}\right)$ and $h(x)=(f\circ g)(x)$.

A suitable domain of $g(x)$ for $h(x)$ to be defined is

A. $[e^{-3}, e^{-1}]$

B. (e^{-3}, e^{-1})

C. $(0, e)$

D. (e^{-2}, e)

E. $(-\infty, e]$

Question 15

x	-2	0	1	5	7
$\Pr(X=x)$	$\frac{1}{8}$	$\frac{1}{7}$	a	b	$\frac{1}{4}$

The table above shows the probability distribution of random variable X which has $\mu=\frac{171}{56}$.

The values of a and b are respectively

A. $\frac{11}{56}$ and $\frac{2}{7}$

B. $\frac{13}{56}$ and $\frac{7}{28}$

C. $\frac{3}{14}$ and $\frac{15}{56}$

D. $\frac{7}{28}$ and $\frac{13}{56}$

E. $\frac{15}{56}$ and $\frac{3}{14}$

Question 16 Four red marbles and four blue marbles are in a bag. Three marbles are drawn at random from the bag. The probability that the three drawn marbles consist of exactly two red marbles is

A. $\frac{1}{4}$

B. $\frac{1}{3}$

C. $\frac{1}{2}$

D. $\frac{5}{8}$

E. $\frac{3}{4}$

Question 17 The standard normal distribution is $N(0,1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and the transformed normal distribution is $N(\mu,\sigma^2)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

The sequence of transformations required to change $N(0,1)$ to $N(-1,4)$ is

A. Dilate vertically by factor $\frac{1}{2}$, dilate horizontally by factor 2, translate to the left by 1

B. Dilate vertically by factor 2, dilate horizontally by factor $\frac{1}{2}$, translate to the right by 1

C. Dilate vertically by factor 2, translate to the left by 1, dilate horizontally by factor $\frac{1}{4}$

D. Translate to the left by 1, dilate vertically by factor $\frac{1}{2}$, dilate horizontally by factor 4

E. Translate to the right by 1, dilate vertically by factor 2, dilate horizontally by factor $\frac{1}{4}$

Question 18 Let X be the number of people having certain attribute in a population. It is a random variable with a normal distribution.

Given $\Pr(X<8000)\approx 0.8$ and $\Pr(X>9000)\approx 0.1$, the mean of X is closest to

A. 5990

B. 6000

C. 6030

D. 6090

E. 6200

Question 19 The proportion of people taller than 180 cm in a large population is 0.25.
Ten random samples of size 5 are taken from the population. Out of the ten random samples the number of samples with two or less people taller than 180 cm is closest to

A. 2

B. 4

C. 5

D. 7

E. 9

Question 20 A random sample of size 3000 was taken from a population to estimate the proportion of diabetic people. It was found that 500 people were diabetic. Another random sample of the same size is to be taken; the probability that it contains 500 diabetic people is closest to

A. 0.15

B. 0.13

C. 0.08

D. 0.04

E. 0.02

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required, an exact value must be given unless otherwise stated.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this examination are **not** drawn to scale.

Question 1 (13 marks)

A playground slide consists of four sections on level ground.
The profile of the slide (from left to right) is given by equations:

Section A $y=\sqrt{x}$; section B $y=\frac{\sqrt{2}}{16}\left(x^2+12\right)$; section C $y=\frac{7\sqrt{2}}{4}$; section D $y=-\frac{7\sqrt{2}}{4}(x-6)$

The sliding section is made up of section A and section B. section C is the platform at the top of the slide.
Section D is the ladder from the ground to the platform.

a. Determine the length of the ladder. 1 mark

b. Calculate the length of the platform. 1 mark

c. Show that section A and section B are joined at $\left(2, \sqrt{2}\right)$. 1 mark

d. Show that section A and section B are joined smoothly. 1 mark

e. Sketch the profile of the slide (all four sections). $y=0$ at ground level. 2 marks

If section A and section B are joined smoothly at $\left(2, \sqrt{2} - \frac{1}{10}\right)$, the equations become $y = \alpha\sqrt{x}$ for section A and $y = ax^2 + b$ for section B.

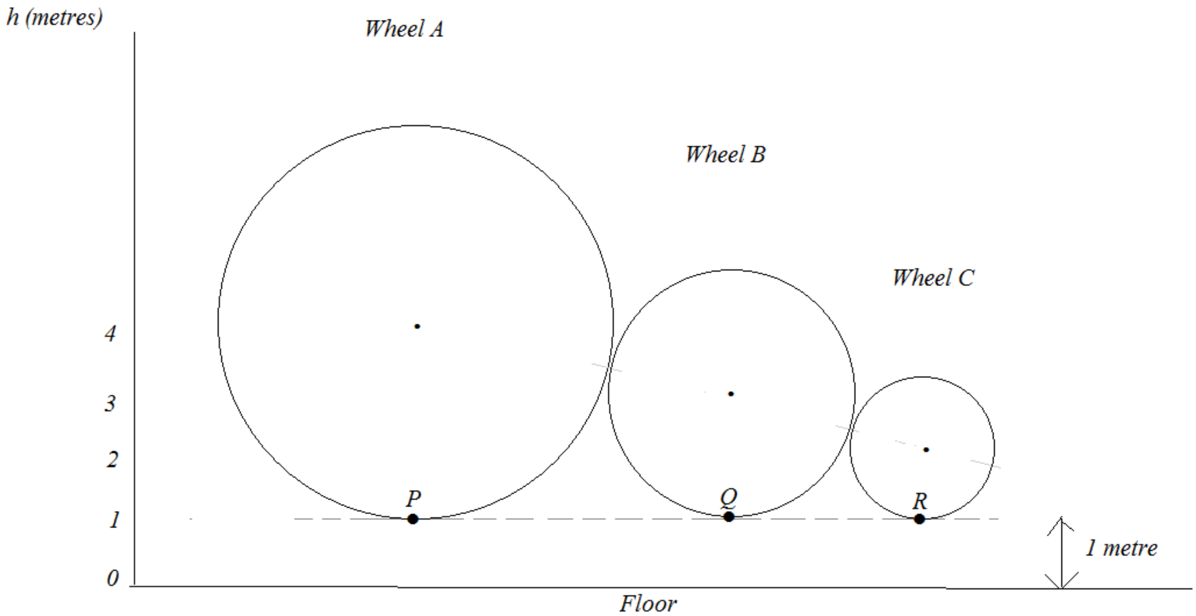
f. Find the value of α, correct to 4 decimal places. 1 mark

g. Find the equation of section B, correct to 4 decimal places. 2 marks

h. The slide is unsafe if there is a point in section B where $y > 2.5$ and gradient > 1.
Determine whether the slide is safe by considering section B only. 2 marks

i. The slide is unsafe if there is a point in section A where $y > 0.45$ and gradient > 1.
Determine whether the slide is safe by considering section A only. 2 marks

Question 2 (13 marks)



Wheel A (radius 3 m), wheel B (radius 2m) and wheel C (radius 1m) are in contact. Each spins about its centre at its own constant speed *without slipping* as shown in the diagram above. The lowest point of each wheel is marked and labeled as P, Q and R respectively. They are 1 metre above the ground initially. Points P, Q and R rise and fall when the wheels spin and their heights above the floor at time t (seconds) are $h_P(t)$, $h_Q(t)$ and $h_R(t)$.

a. Given $h_P(t) = -3\cos(2\pi t) + 4$, find the time for wheel A to complete one revolution. 1 mark

b. Explain and verify that $h_Q(t) = -2\cos(3\pi t) + 3$. 2 marks

c. Explain and verify that $h_R(t) = -\cos(6\pi t) + 2$. 1 mark

d. Points P, Q and R are together at the same level (i.e. 1 metre above the ground) initially.
Find the next two times when they are together again at the same level 1 metre above the ground. 2 marks

e. Find the first time when points P, Q and R are at their maximum heights simultaneously. 2 marks

f. Find the second time when points P, Q and R are at their maximum heights simultaneously. 1 mark

g. Calculate the *rotating speed* (metres per second) of each of points P, Q and R. 1 mark

h. Determine the *maximum rising speed* (metres per second) of each of points P, Q and R.
Either explain or calculate. 1 mark

i. Are points P, Q and R collinear (i.e. on the same line) when they are at their maximum heights? Explain.

2 marks

Question 3 (11 marks)

Consider $\log_e(x)$ and $\log_{10}(x)$.

a. State the transformation required to change $\log_e(x)$ to $\log_{10}(x)$. 1 mark

b. Show that the derivative of $x(\log_e(x)-1)$ is $\log_e(x)$. 1 mark

c. Find the anti-derivative of $\log_e(x)$ such that its value is 1 when $x = e$. 1 mark

d. Show that the derivative of $x\left(\log_{10}(x) - \frac{1}{\log_e(10)}\right)$ is $\log_{10}(x)$. 2 marks

e. Find the anti-derivative of $\log_{10}(x)$ such that its value is 1 when $x = 10$. 1 mark

f. Find $\int_1^a \log_{10}(x)dx$ in terms of a. 1 mark

g. Let A be the area between $y = \log_{10}(x)$, the x-axis and $x = a$. Find the value of a such that $A \geq 10 - \frac{9}{\log_e(10)}$. 2 marks

h. Let B be the area between $y = \log_{10}(x) - 1$, the x-axis and $x = a$. Determine the exact value of $\lim_{a \to 0^+} B$.

2 marks

Question 4 (12 marks)

$f(x) = ke^x(-3+4x-x^2)$ is a probability density function of random variable X, where $k>0$.

a. Show that the minimum value of k is 0.092, correct to 3 decimal places. 2 marks

b. Determine the turning point of $f(x)$ for $k=0.092$, correct to 3 decimal places. 1 mark

c. Sketch the graph of probability density function $f(x)$ for $k=0.092$. Show the important features. Comment on the difference between $f(x)$ and a normal distribution.

3 marks

d. Determine the mean μ and standard deviation σ of random variable X, correct to 2 decimal places.

2 marks

e. Evaluate $\Pr(\mu - 2\sigma < X < \mu + 2\sigma)$. 1 mark

f. For $g(x) = 0.05e^{x}\left(c + 6x - x^{2}\right)$ to be a probability density function, $c > 9$. Explain why c is greater than 9.

1 mark

g. Consider $h(x) = 0.05e^{x}(x-2)(b-x)$ where $b > 2$.
Find the maximum value of b if $h(x)$ is a probability density function, correct to 2 decimal places. 2 marks

Question 5 (11 marks)

A two-year-old survey showed that 25% of the families in a country lived below the poverty line.
To determine if this percentage has changed, three random samples of different sample size n are taken.
Random sample A consists of 50 families and 8 are found to be living below the poverty line.
Random sample B consists of 100 families and 16 are below the poverty line.
Random sample C consists of 250 families and 40 are below the poverty line.

a. Calculate the sample proportion $\hat{p}$ of families living below the poverty line for each sample.
Use each sample to estimate the population proportion p and standard deviation σ in the country living below the poverty line.
Calculate the interval $\left(\hat{p} - 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$ in each case. 5 marks

b. Which one of the three random samples gives the best indication whether the percentage of the families in the country lived below the poverty line has changed? Is the percentage increased, the same or decreased?

2 marks

c. By calculation find the *best* estimation of the probability in getting each of the three random samples.

3 marks

d. Comment on the relation between probability found in part c and the reliability of the random sample in determining whether the percentage of the families in the country lived below the poverty line has changed.

1 mark

End of Examination 2

