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2024 Mathematical Methods

Year 12

Application Task

Time allowed: 4 hours plus

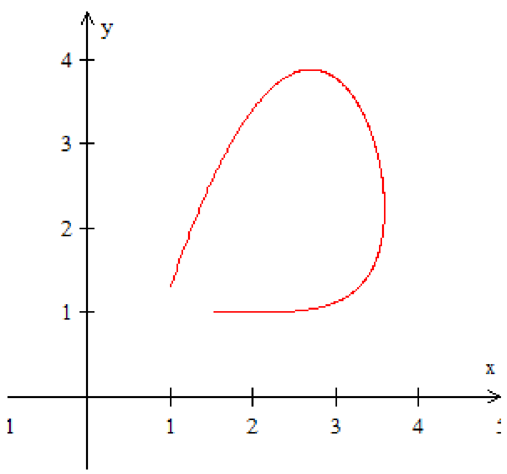
Application Task

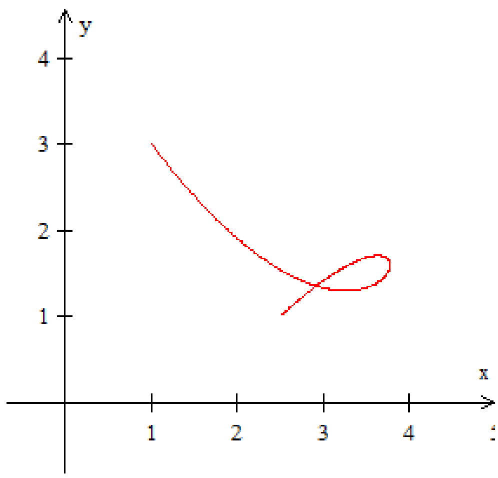
Theme: Binomial expansion, Bernstein polynomials and Bézier curves

Assumed knowledge:

Functions, relations, graphs, binomial expansion, transformations, co-ordinates, parameters, combination ${}^{n}C_{i}$, differentiation, calculus, CAS, graphing parametric equations by CAS

Introduction:
A Bernstein polynomial is a polynomial that is a linear combination of the terms in the binomial expansion $(p+q)^n$ where $n = 0, 1, 2, 3, \ldots$ and $p = 1 - q$.
By restricting $p \in [0, 1]$, Bernstein polynomials can be used to generate Bézier curves to model smooth curves. The following graphs are examples of Bézier curves.





Part I (80 minutes plus)

Investigation of Binomial expansion

a. Write down the expansion of each of the following expressions.

$(p+q)^0 =$

$(p+q)^1 =$

$(p+q)^2 =$

$(p+q)^3 =$

$(p+q)^4 =$

b. Evaluate the following combinations. Note: $\binom{n}{i} = {}^{n}C_{i}$

$\binom{0}{0} =$ $\binom{1}{0} =$ $\binom{1}{1} =$ $\binom{2}{0} =$ $\binom{2}{1} =$ $\binom{2}{2} =$

$\binom{3}{0} =$ $\binom{3}{1} =$ $\binom{3}{2} =$ $\binom{3}{3} =$

c. Write down the coefficients of p^4, p^3q, p^2q^2, pq^3, q^4 in the expansion of $(p+q)^4$ using $\binom{n}{i}$ notation.

d. Consider $n > 7$ and n is even.
Write down the **first** three terms, the **middle** term and the **last** three terms in the expansion of $(p+q)^n$. Express your answers in terms of p, q and n.

e. Given $(p+q)^n = 1$, express p in terms of q.

f. Show that $\binom{n}{i} = \binom{n}{n-i}$.

g. Evaluate $\dfrac{\binom{2021}{0}+\binom{2021}{1}+\binom{2021}{2}+\binom{2021}{3}+\cdots\cdots+\binom{2021}{2018}+\binom{2021}{2019}+\binom{2021}{2020}+\binom{2021}{2021}}{\binom{2020}{0}+\binom{2020}{1}+\binom{2020}{2}+\binom{2020}{3}+\cdots\cdots+\binom{2020}{2017}+\binom{2020}{2018}+\binom{2020}{2019}+\binom{2020}{2020}}$.

In the following parts, consider cubic binomial, i.e. $n = 3$ and $S_{cubic} = (p+q)^3$.
Let $q = t$ and $p = 1-t$ where $t \in [0, 1]$.

h. In terms of t, the third term in the expansion of $(p+q)^3$ is $T_3 = 3(1-t)t^2$.
In terms of t, write down the first, second and fourth terms of S_{cubic}, i.e. T_1, T_2 and T_4.

i. Sketch the graphs of $T_1(t), T_2(t), T_3(t), T_4(t)$ and S on the same axes. Label each graph.

j. State the maximum and minimum values of each of T_1 and T_4.
Find the maximum and minimum values of each of T_2 and T_3 by calculus.

k. Evaluate $T_1 + T_2 + T_3 + T_4$. Explain your answer.

End of Part I

Part II (80 minutes plus)

In Part I, you considered the terms in the expansion of cubic ($n=3$) binomial $S_{cubic}=(p+q)^3$ where $q=t$, $p=1-t$ and $t\in[0,1]$.
The terms are labeled as T_1, T_2, T_3 and T_4, e.g. $T_3=3(1-t)t^2$.

In Part II, you investigate and use **Bernstein polynomials** to fit curves.
A Bernstein polynomial is a polynomial that is a **linear combination** of the terms $T_1, T_2, T_3, \cdots$ and T_n in the binomial expansion $(p+q)^n$ where $n=0, 1, 2, 3, \ldots$, $q=t$ and $p=1-t$.

An example of a **cubic** Bernstein polynomial is $1.2T_1+5T_2-2T_3+7T_4$,
i.e. $1.2\times(1-t)^3+5\times 3(1-t)^2 t-2\times 3(1-t)t^2+7\times t^3$.
In general, a cubic Bernstein polynomial has the form $aT_1+bT_2+cT_3+dT_4$ where a, b, c and $d\in R$. Factor a causes a 'vertical' dilation of T_1, factor b causes a vertical dilation of T_2 etc.

A general quartic Bernstein polynomial is $aT_1+bT_2+cT_3+dT_4+eT_5$.

Now consider the expansion of quartic ($n=4$) binomial $S_{quartic}=(p+q)^4$ where $q=t$, $p=1-t$ and $t\in[0,1]$. The middle term is $T_3=6(1-t)^2t^2$.

a. In terms of t, write down the other four terms in the expansion of $S_{quartic}=(p+q)^4$, i.e. terms T_1, T_2, T_4 and T_5.

b. Select appropriate real values of a, b, c, d and e in the interval $[-5, 3]$, write a **quartic** Bernstein polynomial in terms of t.

In the following parts, you investigate only **cubic** Bernstein polynomials of the form $aT_1 + bT_2 + cT_3 + dT_4$.

If $a = b = c = d = 1$, the cubic Bernstein polynomial is simply $S_{cubic} = (1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3$.

c. Let $b = c = d = 1$ and choose a value of $a \in [2, 5]$.

Sketch the graph of the cubic Bernstein polynomial of your chosen value of a and the graph of S_{cubic} for $t \in [0, 1]$ on the same axes.

Compare and state the effects of changing the value of a on the graph of S_{cubic}, particularly on the endpoints, minimum and maximum values of the polynomial.

d. Let $a = b = c = 1$ and choose a value of $d \in [-2, 0)$.

Sketch the graph of the cubic Bernstein polynomial of your chosen value of d for $t \in [0, 1]$.

State the effects of changing the value of d on the graph of S_{cubic}, particularly on the endpoints, minimum and maximum values of the polynomial.

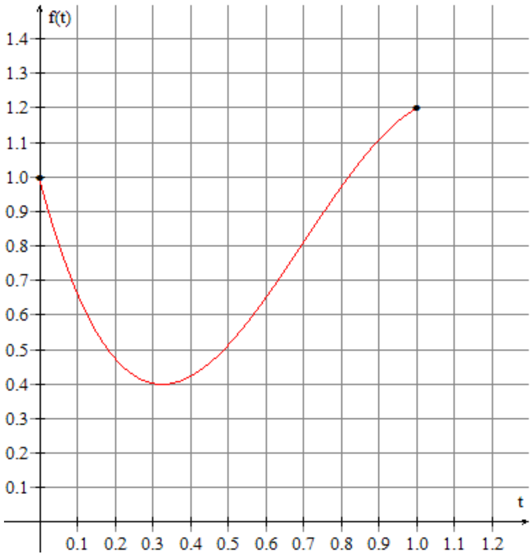
e. Let $a = c = d = 1$ and choose two values of b, one from each interval $[-2, 1)$ and $(1, 2]$. Sketch the graph of the cubic Bernstein polynomial for each of your chosen values of b for $t \in [0, 1]$.
State the effects of changing the value of b on the graph of S_{cubic}, particularly on the endpoints, minimum and maximum values of the polynomial.

f. Let $a = b = d = 1$. Choose two values of c, one from each interval $[-5, -2]$ and $(1, 2]$. Sketch the graph of the cubic Bernstein polynomial for each of your chosen values of c for $t \in [0, 1]$.
State the effects of changing the value of c on the graph of S_{cubic}, particularly on the endpoints, minimum and maximum values of the polynomial.

g. For S_{cubic}, $a = b = c = d = 1$. If you want to change the right endpoint of S_{cubic}, which one/more of a, b, c or d do you change?

h. There are ways that you can change S_{cubic} so that it has maximum and/or minimum values as well as turning points in the interval $[0, 1]$ after the change.
State two ways to change S_{cubic}.

i. The graph of a cubic Bernstein polynomial $f(t) = a(1-t)^3 + b3(1-t)^2 t + c3(1-t)t^2 + d\,t^3$ is shown below.



Use your CAS to determine the value of each of a, b, c and d, correct to 1 decimal place.
Hint: Start from the graph of $S_{cubic} = (1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3$, vary the values of a, b, c and d if necessary according to your findings in previous parts.

j. By calculus show that the minimum value of $f(t)$ is approximately 0.4.
Determine the corresponding value of t, correct to 1 decimal place.

End of Part II

Part III (80 minutes plus)

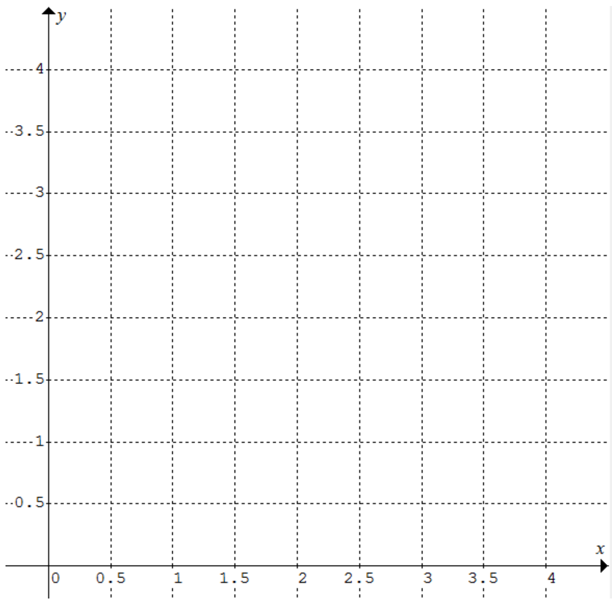
In Part II, you investigated cubic Bernstein polynomials.
In Part III, you study and sketch simple **Bézier curves**.
A **Bézier curve** is a parametric curve with co-ordinates $x(t)$ and $y(t)$ given by Bernstein polynomials, $t \in [0, 1]$.
Here, you use only **cubic** Bernstein polynomials
$x(t) = a(1-t)^3 + b3(1-t)^2 t + c3(1-t)t^2 + d\,t^3$ and
$y(t) = e(1-t)^3 + f\,3(1-t)^2 t + g3(1-t)t^2 + ht^3$ where a, b, c, d, e, f, g and $h \in R$.
The resulting curve is a **cubic** Bézier curve.

You can sketch a simple Bézier curve by plotting, or use CAS to display a curve.
Using the TI-Nspire: **Graphs** application (**New Document** > **Add Graphs**), menu > **Graph Entry/Edit** > **Parametric**, enter parametric equations and $0 \le t \le 1$.

a. A Bézier curve is defined by $x(t) = (1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + 4t^3$ and
$y(t) = 2(1-t)^3 + \frac{1}{3} \times 3(1-t)^2 t + (-1) \times 3(1-t)t^2 + 3t^3$.
Fill in the missing entries to complete the following table for the coordinates of 11 points on the curve.

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$x(t)$	1	1.003	1.024	1.081	1.192	1.375		2.029	2.536	3.187	4
$y(t)$	2	1.515	1.080	0.725	0.480	0.375		0.705	1.200	1.955	3

b. Plot the points and draw a smooth curve through the points.



c. Endpoints of a cubic Bézier curve are (a, e) and (d, h) where a, d, e and h are defined in the introduction to Part III.
State the value of each of a, d, e and h for the cubic Bézier curve in part a.

Further information: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

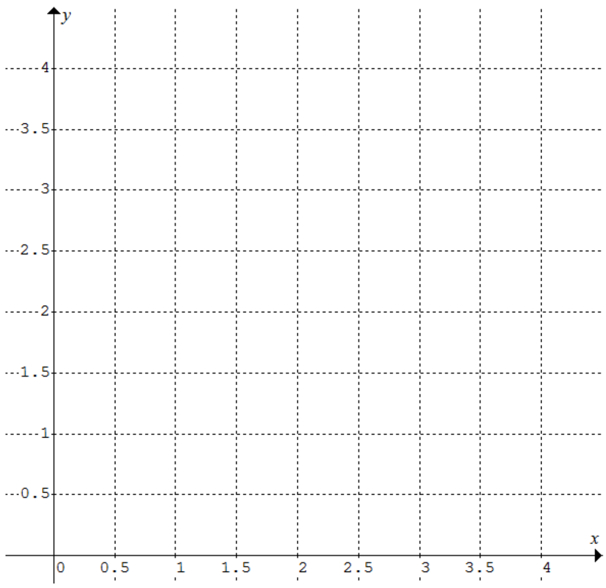
d. Determine the co-ordinates of the turning point of the cubic Bézier curve in part a, correct to 3 decimal places. Show working.

Consider the cubic Bézier curve given by the following parametric equations:

$x(t) = (1-t)^3 + 4\times3(1-t)^2 t + 5\times3(1-t)t^2 + 2t^3$ and

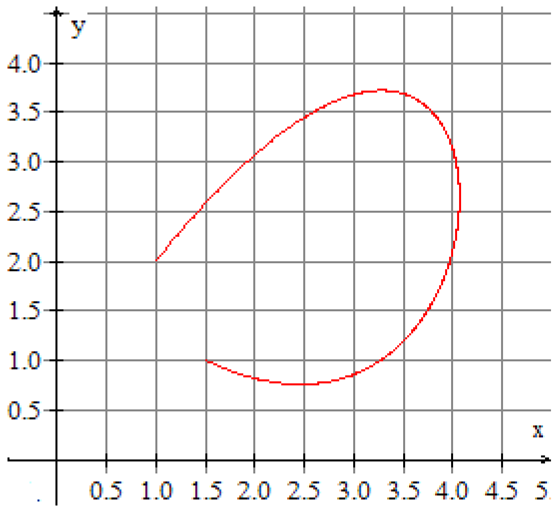
$y(t) = 3(1-t)^3 - \frac{1}{6}\times3(1-t)^2 t + 4\times3(1-t)t^2 + t^3$

e. With the help of CAS, sketch the cubic Bézier curve given above.



f. State the domain and range of the cubic Bézier curve given above using inclusive brackets [,], in exact form if possible, otherwise correct to 3 decimal places.

Consider the cubic Bézier curve given by the following graph:



The parametric equations are: $x(t) = a(1-t)^3 + b3(1-t)^2 t + c3(1-t)t^2 + dt^3$ and $y(t) = e(1-t)^3 + f3(1-t)^2 t + g3(1-t)t^2 + ht^3$ where a, b, c, d, e, f, g and $h \in R$.

The graph is a smooth curve with two endpoints $(1.000, 2.000)$ and $(1.500, 1.000)$.
The maximum point is $(3.2754, 3.7189)$ corrected to 4 decimal places.
Also, assume $c = 5.000$ and $f = 7.000$ in the parametric equations.

g. Explain why $a = 1.000$, $d = 1.500$, $e = 2.000$ and $h = 1.000$ in the parametric equations.

h. Use the maximum point $(3.2754, 3.7189)$ to show that $g = -0.667$ in the parametric equation for $y(t)$, $b = 5.000$ in the parametric equation for $x(t)$, and parameter $t = 0.2529$ at the maximum point .

i. Find the co-ordinates of the minimum point, correct to 3 decimal places.

j. For the Bezier curve with parametric equations:
$x(t) = a(1-t)^3 + b3(1-t)^2 t + c3(1-t)t^2 + d\,t^3$ and
$y(t) = e(1-t)^3 + f\,3(1-t)^2 t + g3(1-t)t^2 + h\,t^3$,
find the gradient of the curve at each end point in terms of a, b, c, d, e, f, g and $h \in R$.
Comment.

k. Select a suitable value $m \in [-0.450, -0.150]$.
Find the co-ordinates of two points on the curve having the same gradient m of your selected value.

End of Part III
End of Application Task

