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2024 Mathematical Methods

Year 12

Modelling Task

Time allowed: 2 hours plus

Modelling Task

Theme: Ocean waves

Ocean waves change shape as they travel towards shore. The waves break when they are close to shore. Two photographs of breaking waves are shown below.

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Assumed knowledge: Relations and functions, sine and cosine functions, transformations, simultaneous equations, calculus, gradient, area and volume, definite integral, graphs, CAS

Information/specification:

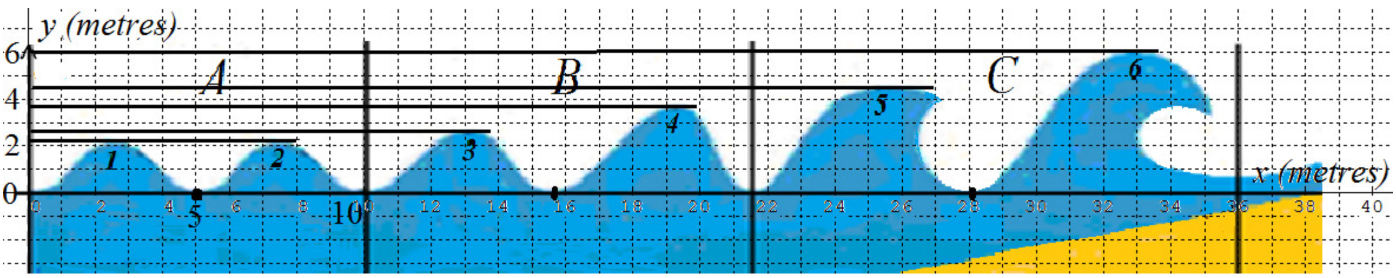
The following graph (**NOT drawn to scale**) shows the cross-section of surface ocean waves approacing shore at a particular moment. The graph is divided into sections *A*, *B* and *C*.

In Section *A* , Waves 1 and 2 are identical. They are furthest from shore.

In Section B, Wave 4 is the transformation of Wave 3 under a dilation from the *x*-axis.

In Section C, Waves 5 and 6 are breaking waves. Their cross-sections are different.

All length (distance) measurements are in metres, area in square metres, and volume in cubic metres. ***Correct your answers to two decimal places*** unless stated otherwise.



Part I (70 minutes plus)

a. The height of Waves 1 and 2 in Section *A* above the *x*-axis is 2.20 m. The lowest points touch the *x*-axis.

Two cycles of $y = a\cos(bx) + c$ are used to model the wave outline (at and above the *x*-axis) in Section *A*.

Find the values of a, b and $c \in R$. State the domain of $y = a\cos(bx) + c$.

b. $y = \alpha\sin(\beta x + \gamma) + c$ can also be used to model the wave outline in Section *A*.

Find the values of α, β and $\gamma \in R$.

c. Determine the area enclosed by the wave outline in Section *A* and the *x*-axis.

$(10, 0)$, $(15.75, 0)$ and $(21.5, 0)$ are the lowest points (turning points) of Waves 3 and 4 in Section *B*, whilst $(13.35, 2.7)$ and $(19.1, 3.7)$ are the highest points (turning points).

d. Cubic polynomial $y = px^3 + qx^2 + rx + s$ is used to model the outline of Wave 3 from $(10, 0)$ to $(13.35, 2.7)$.

Set up and solve four simultaneous equations for coefficients p, q, r and $s \in R$ to show that $p \approx -0.143635$, $q \approx 5.0308$, $r \approx -57.5257$ and $s \approx 215.8111$.

The equation of the outline of Wave 3 from $(13.35, 2.7)$ to $(15.75, 0)$ is obtained by three transformations (stated below) of equation
$y = -0.143635x^3 + 5.0308x^2 - 57.5257x + 215.8111$.

Transformation 1: Reflection in the *y*-axis.

Transformation 2: Dilation from the *y*-axis by a factor of k.

Transformation 3: Translation of u units in the positive *x*-direction.

e. Write down the transformed equation of
$y = -0.143635x^3 + 5.0308x^2 - 57.5257x + 215.8111$ after Transformation 1.

f. Investigate the effects on the modelling curve when the number of decimal places of the coefficient of x^3 is reduced.

g. Show that the dilation factor k in Transformation 2 is $\frac{48}{67}$.

h. Show that $u \approx 22.9142$ in Transformation 3.

i. Show that, after applying Transformation 1, Transformation 2 and Transformation 3 to $y = -0.143635x^3 + 5.0308x^2 - 57.5257x + 215.8111$, the transformed equation is
$y \approx 0.390626(x - 22.9142)^3 + 9.8018(x - 22.9142)^2 + 80.2963(x - 22.9142) + 215.8111$

j. Verify that $y = -0.143635x^3 + 5.0308x^2 - 57.5257x + 215.8111$ and $y \approx 0.390626(x - 22.9142)^3 + 9.8018(x - 22.9142)^2 + 80.2963(x - 22.9142) + 215.8111$ are **joined smoothly**, correct to 2 decimal places.

k. The equation of the outline of Wave 3 is $y = f(x)$.

Express $f(x)$ as a piecewise function with the coefficients of x^3 corrected to 6 decimal places and the other coefficients to 4 decimal places.

l. Determine the greatest positive gradient of the outline of Wave 3.

Hence determine the greatest negative gradient of the outline of Wave 3.

(Clarification: As an example, the **greatest** among the negative gradients -1.5, -3.8 and -2.6 is -3.8)

m. Determine the cross-sectional area of Wave 3 above the *x*-axis.

Hence find the volume of water of Wave 3 above the *x*-axis for two metres of the wave, assuming uniform cross-section throughout the two metres.

End of Part I

Information/specification:

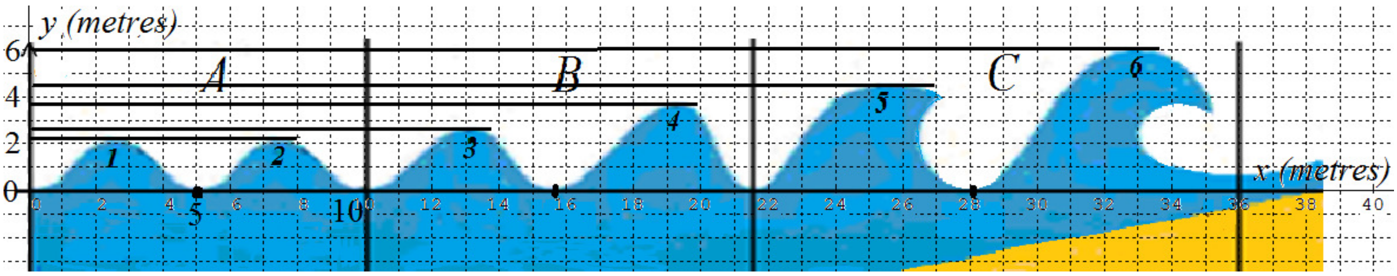
The following graph (**NOT drawn to scale**) shows the cross-section of surface ocean waves approacing shore at a particular moment. The graph is divided into sections A, B and C.

In Section A , Waves 1 and 2 are identical. They are furthest from shore.

In Section B, Wave 4 is the transformation of Wave 3 under a dilation from the x-axis.

In Section C, Waves 5 and 6 are breaking waves. Their cross-sections are different.

All length (distance) measurements are in metres, area in square metres, and volume in cubic metres. ***Correct your answers to two decimal places*** unless stated otherwise.



Part II (70 minutes plus)

The surface of Wave 3 in Section B is modelled by

$y = -0.143635x^3 + 5.0308x^2 - 57.5257x + 215.8111$ and

$y \approx 0.390626(x-22.9142)^3 + 9.8018(x-22.9142)^2 + 80.2963(x-22.9142) + 215.8111$.

$(10,\ 0), (15.75,\ 0)$ and $(21.5,\ 0)$ are the lowest points (turning points) of Waves 3 and 4, whilst $(13.35,\ 2.7)$ and $(19.1,\ 3.7)$ are the highest points (turning points).

Wave 4 is the transformation of Wave 3 under a dilation by a factor of h from the x-axis, and then a translation of w units in the positive x-direction.

a. Show that $h \approx 1.3704$ and $w = 5.75$.

b. Wave 4 has equation $y = f(x)$. Express $f(x)$ as a piece-wise function with the coefficient of x^3 corrected to 6 decimal places and the other coefficients to 4 decimal places.

c. Determine the value of each of the following ratios.

(Clarification: As an example, the **greatest** among the negative gradients -1.5, -3.8 and -2.6 is -3.8)

Ratio 1: $\dfrac{\text{greatest positive gradient of the surface of Wave 4}}{\text{greatest positive gradient of the surface of Wave 3}}$

Ratio 2: $\dfrac{\text{greatest } \textit{negative} \text{ gradient of the surface of Wave 4}}{\text{greatest } \textit{positive} \text{ gradient of the surface of Wave 3}}$

d. Determine the value of the ratio $\dfrac{\text{cross-sectional area above the } x\text{-axis of Wave 4}}{\text{cross-sectional area above the } x\text{-axis of Wave 3}}$.

Now consider Wave 5, $21.5 \le x \le 28.1$.

The upper outline of Wave 5 in Section C is modelled by the quartic equation

$y = 0.0278518x^4 - 2.7587x^3 + 101.9787x^2 - 1666.6577x + 10159.2199$, and the lower outline is modelled by an **arc** of the circle $(x - 28.10)^2 + (y - R)^2 = R^2$ with radius R.

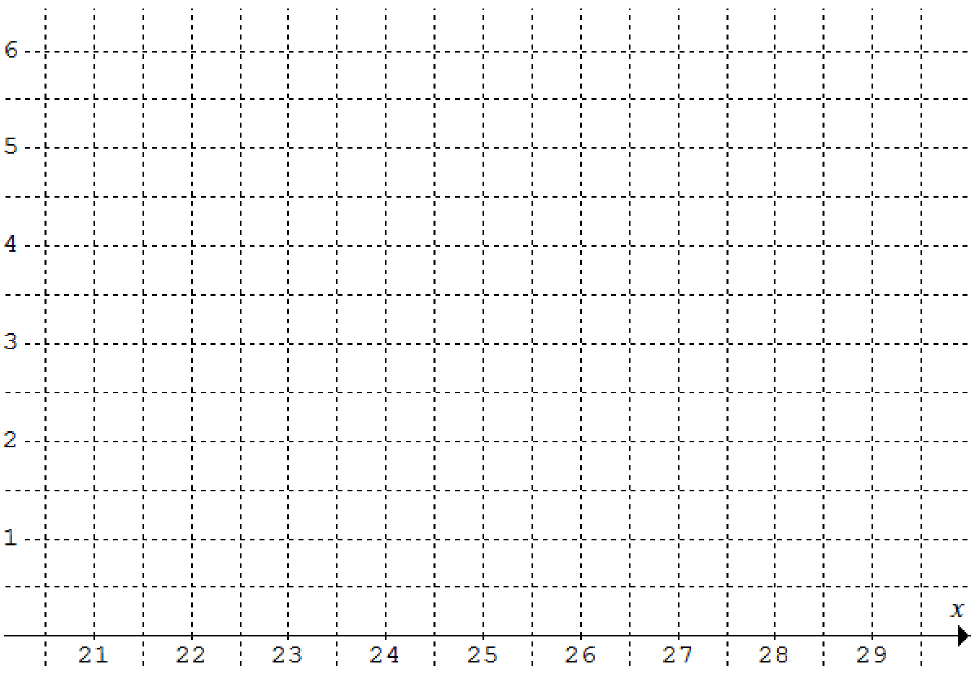
The tip of Wave 5 is the point $(27.10, 4.00)$ where the quartic curve and the circular arc intersect.

e. Show that the radius $R = 2.125$.

f. Determine the domain of the **circular arc** modelling the lower ouline of Wave 5.

g. Sketch Wave 5 (upper and lower outlines) in the following grid.

Show coordinates (correct to 2 decimal places) of typical points which specify Wave 5.



Consider Wave 6 in Section C.

Information about the upper outline of Wave 6:
$(28.1, 0)$ and $(33.1, 6)$ are turning points.
$(35.1, 3.3)$ is the tip of the wave.

h. Using only turning points $(28.1, 0)$ and $(33.1, 6)$, a student models the upper outline of Wave 6 with equation $y = -3\cos(n(x - 28.1)) + 3$ where $n \approx 0.6283$.
Explain and show workings in finding $n \approx 0.6283$.

i. Show that curve $y \approx -3\cos(0.6283(x - 28.1)) + 3$ does not end at the tip of the wave $(35.1, 3.3)$.

j. Instead of using the pair of turning points $(28.1, 0)$ and $(33.1, 6)$, the student makes a slight change of the second turning point from $(33.1, 6)$ to $(\tau, 6)$, then use $(28.1, 0)$ and $(\tau, 6)$ to work out an alternative model $y = -3\cos(\eta(x - 28.1)) + 3$ which ends at the tip of the wave $(35.1, 3.3)$.
Determine the values of η and τ, correct to 4 decimal places.

k. Discuss the limitations of using cosine function such as $y = -3\cos(n(x - 28.1)) + 3$ to model the upper outline of Wave 6.

l. Choose a polynomial function to model the upper outline of Wave 6, which could satisfy the requirements that $(28.1, 0)$ and $(33.1, 6)$ are turning points, and $(35.1, 3.3)$ is the tip of the wave. Give reasons to support your choice.

End of Part II

End of Modelling Task