2024 NSW ESA Mathematics Advanced Solutions

Section I

1	2	3	4	5	6	7	8	9	10
C	B	A	C	A	D	C	D	B	B

Q1 Positive y-intercept, negative gradient C

Q2 $60-5=55$ play
Let x students play both, $(38-x)+x+(35-x)=55$ $\therefore$ $x=18$ B

Q3 English: 2σ from μ; the others are less than 2σ. A

Q4 $y=(x-3)^2-2 \to y=(-x-3)^2-2 \to -y=(-x-3)^2-2$
$\therefore$ $y=-(x+3)^2+2$ C

Q5 $\int (6x+1)^3\,dx=\dfrac{(6x+1)^4}{6\times 4}+c$ A

Q6 $x^2-1>0$, $x^2>1$ $\therefore$ $x<-1$ or $x>1$ D

Q7 Translate to the right by 1 and then dilate by $\dfrac{1}{2}$ horizontally. C

Q8 Box plot shows wide inter-quartile range D

Q9 $\Pr(\text{both red}\,|\,\text{at least one red})=\dfrac{\frac{2}{5}\times\frac{1}{4}}{1-\frac{3}{5}\times\frac{2}{4}}=\dfrac{1}{7}$ B

Q10 $f(x)=\dfrac{dA}{dx}$, $\dfrac{d^2A}{dx^2}=f'(x)=0$, 3 points of inflection. B

Section II

Q11

x-value	$g'(x)$	$g''(x)$
$x=-3$	*positive*	*negative*
$x=1$	*zero*	*zero*
$x=5$	*positive*	*positive*

Q12 Number of terms $n=\dfrac{2024-50}{7}+1=283$, $a=50$, $\ell=2024$

Sum of terms $=\dfrac{n}{2}(a+\ell)=293471$

Q13

	W	K
1985 population	$A=34$	$B=280$
% yearly change	+5.5%	–3%
Population when $x=50$	$34\times 1.055^{50}\approx 494$	61

Q14a $(x-1)^2=5-x^2$, $2x^2-2x-4=0$, $2(x-2)(x+1)=0$
$\therefore$ $x=-1,\ 2$

Q14b Area $=\displaystyle\int_{-1}^{2}\left(\left(5-x^2\right)-(x-1)^2\right)dx=\int_{-1}^{2}\left(-2x^2+2x+4\right)dx$

$=\left[-\dfrac{2x^3}{3}+x^2+4x\right]_{-1}^{2}=9$

Q15 $\dfrac{dV}{dt}=300-7.5t=0$ $\therefore$ $t=40$

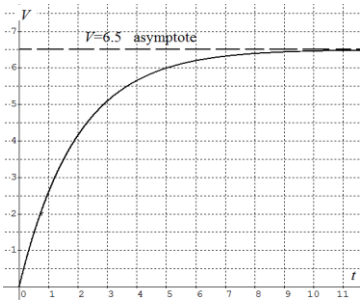
$\Delta V=\displaystyle\int_0^{40}(300-7.5t)dt=\left[300t-\frac{7.5}{2}t^2\right]_0^{40}=6000$

$V(40)=6000+350=6350$

Q16 A is negatively skewed and B is positively skewed.
The median of A is higher than that of B.
A has a wider spread than B.

	A	*B*
Skewness	*negative*	*positive*
Central tendency	*approx. 171*	*approx. 152*
Spread (range)	*approx. 40*	*approx. 28*

Q17a



Q17b When $t=1$, $2.6=6.5\left(1-e^{-k}\right)$, $e^{-k}=0.6$, $e^{k}=\dfrac{1}{0.6}$

$k=\ln\dfrac{1}{0.6}\approx 0.511$

Q17c When $t=2$, $\dfrac{dV}{dt}=6.5ke^{-kt}=6.5k\left(e^{-k}\right)^t$

$\approx 6.5\times 0.511\times 0.6^2\approx 1.196$ volts per second

Q18a $(1-0.15)^2=0.85^2=0.7225$

Q18b $\Pr(\text{at least one})=1-\Pr(\text{none})=1-0.85^n>0.8$ $\therefore$ $n=10$

Q19 $y=x^4-2x^3+2$, $\dfrac{dy}{dx}=4x^3-6x^2=2x^2(2x-3)=0$

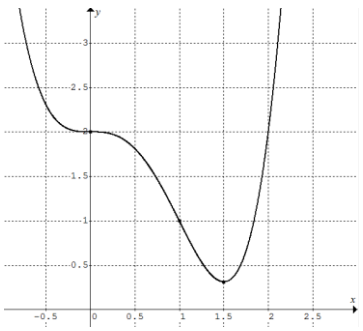
$\therefore$ $x=0$ and $y=2$; $x=\dfrac{3}{2}$ and $y=\dfrac{5}{16}$

	$x<0$	$x>0$	$x<3/2$	$x>3/2$
dy/dx	*Negative*	*Negative*	*Negative*	*Positive*
Nature	$(0,\ 2)$ *stationary inflection point*		$(3/2,\ 5/16)$ *minimum turning point*	

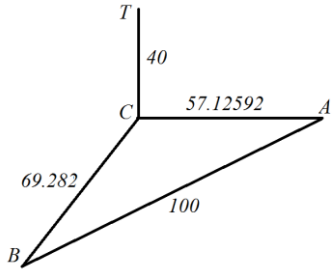
Stationary points are $(0,\ 2)$ and $\left(\dfrac{3}{2},\dfrac{5}{16}\right)$.

$\dfrac{d^2y}{dx^2}=12x^2-12x=12x(x-1)=0$ $\therefore x=0, y=2$; $x=1$, $y=1$

Points of inflection are $(0,\ 2)$ and $(1,1)$.



Q20a $\tan 35° = \dfrac{40}{AC}$, $AC \approx 57.13$



Q20b $\tan 30° = \dfrac{40}{BC}$, $BC \approx 6.9282$,

$\cos \angle C \approx \dfrac{69.282^2 + 57.126^2 - 100^2}{2 \times 69.282 \times 57.126}$, $\angle C \approx \cos^{-1}(-0.24466) \approx 104°$

Bearing of B from C $\approx 104° + 90° = 194°$

Q21 Observation 1: Both males and females grow faster before maturity.
Observation 2: Both males and females have a linear growth before maturity.
Observation 3: Females are longer than males.
Observation 4: Little difference in length at birth between the two. The gap widens with age because the males have a slow growth after maturity whilst the females continue a linear growth at a lower pace.

Q22a For $-1 < x < 0$, $f'(x) = \dfrac{2x}{1+x^2} < 0$. For $x = 0$, $f'(x) = 0$.
For $0 < x < 1$, $f'(x) > 0$ $\therefore$ concave up

Q22b Estimated area by using trapezoidal rule

$= 2\left(\dfrac{1}{2} \times 0.25\right) \times$

$\big((0 + 0.0606) + (0.0606 + 0.2231) + (0.2231 + 0.4463)$

$+ (0.4463 + 0.6931)\big) \approx 0.5383$

Q22c Overestimate, the graph is concave up and the top side of each trapezium is above the curve.

Q23a $z = \dfrac{70 - 58}{15} = 0.8$ $\Pr(z < 0.8) = 0.7881$

$\therefore \Pr(58 < X < 70) = 0.7881 - 0.5 = 0.2881$ $\therefore$ 28.81%

Q23b 46 and 70 are the same 'distance' from the mean 58 on each side of the mean of a normal distribution which is symmetrical about the mean.

Q23c Top 10%, $\Pr(X < x) = 0.9$, $\Pr(z < 1.3) \approx 0.9$

$\therefore \dfrac{x - 58}{15} \approx 1.3$, $x \approx 77.5$ Since $0.9 < 0.9032$ $\therefore x \approx 77$

Q24a Interest rate per month $= \dfrac{6\%}{12} = 0.005$

End of first month 80×1.005

End of second month $(80 \times 1.005 + 80)1.005$

End of third month $((80 \times 1.005 + 80)1.005)1.005$

$\vdots$

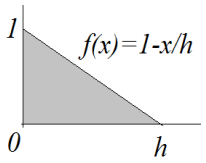
End of 24th month (after simplifying)

$80\left(1.005^{24} + 1.005^{23} + 1.005^{22} + \cdots + 1.005\right) = \dfrac{80 \times 1.005\left(1.005^{24} - 1\right)}{1.005 - 1}$

≈ 2044.73

Q24b $A = \dfrac{2044.73}{80} \approx 25.559$

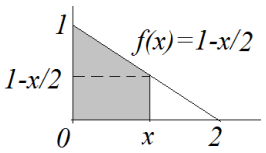
Q25a Shaded area $= \dfrac{1}{2} \times h \times 1 = 1$ $\therefore h = 2$

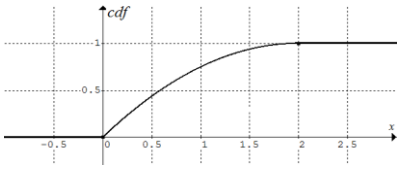


Q25b

cdf = area of shaded area

$= \dfrac{1}{2}\left(1 + \left(1 - \dfrac{x}{2}\right)\right)x = x - \dfrac{x^2}{4}$, $0 \le x \le 2$





Q25c Let m be the median. At $x = m$, $cdf = m - \dfrac{m^2}{4} = \dfrac{1}{2}$

$\therefore m^2 - 4m + 2 = 0$ $\therefore m \approx 0.586$

Q26 2.4% per year $= \dfrac{2.4\%}{12} = 0.002$

Start of last 12 years, amount $= 1200 \times 106.592 = 127910.40$

For this amount, let x be the amount required at the very beginning (180 months ago), $x \times 1.002^{180} = 127910.40$,

$x = 89272.147$

Start of the first 15 years, amount $= 2000 \times 151.036 = 302072.00$

Minimum amount $= 89272.15 + 302072.00 = 391344.15$

Q27a $\dfrac{d}{dx}\left(x^2 \tan x\right) = x^2 \sec^2 x + 2x \tan x$

$= x^2\left(1 + \tan^2 x\right) + 2x \tan x = x^2 + x^2 \tan^2 x + 2x \tan x$

Q27b $\displaystyle\int (x \tan x + 1)^2 dx = \int \left(x^2 \tan^2 x + 2x \tan x + 1\right) dx$

$\displaystyle= \int \left(x^2 + x^2 \tan^2 x + 2x \tan x + 1 - x^2\right) dx$

$\displaystyle= \int \left(\left(x^2 + x^2 \tan^2 x + 2x \tan x\right) + \left(1 - x^2\right)\right) dx = x^2 \tan x + x - \frac{x^3}{3} + c$

Q28a Amplitude $= k = \dfrac{39 - 3}{2} = 18$

$A = 3$ when $t = 0$ $\therefore c - k = 3$ $\therefore c = 21$

Q28b Period $T = \dfrac{2\pi}{\frac{\pi}{24}} = 48$

Q28c Same heights $B(t) = A(t)$ $\therefore \cos\left(\dfrac{\pi}{24}(t - 6)\right) = \cos\left(\dfrac{\pi}{24}t\right)$

$\dfrac{\pi}{24}(t - 6) = \pm\dfrac{\pi}{24}t$, choose $\dfrac{\pi}{24}(t - 6) = -\dfrac{\pi}{24}t$, $t = 3$

$\therefore B(3) = A(3) = 21 - 18\cos\dfrac{\pi}{8} \approx 4.37$

$\dfrac{1}{2}T$ later, $t = 3 + 24 = 27$, $B(27) = A(27) = 21 - 18\cos\dfrac{7\pi}{8} \approx 37.63$

Q29 $y = ax^2 + bx + c$, $\frac{dy}{dx} = 2ax + b$

At (x, y), tangent $t(x) = 2x + 3$, normal $n(x) = -\frac{1}{2}x - 2$.

$\therefore\ 2x + 3 = -\frac{1}{2}x - 2\ \therefore\ x = -2$ and $y = -1\ \therefore\ (-2, -1)$

$\frac{dy}{dx} = 2ax + b = -4a + b = 2$

At $x = -4$, $\frac{dy}{dx} = 2ax + b = 0\ \therefore\ -8a + b = 0$

Solve for $a = \frac{1}{2}$, $b = 4$ and $y = \frac{1}{2}(-2)^2 + 4(-2) + c = -1\ \therefore\ c = 5$

Q30 $\frac{1 - x^{n+1}}{1 - x} = \frac{(1 - x)(1 + x + x^2 + \cdots + x^n)}{1 - x} = 1 + x + x^2 + \cdots + x^n$

$\therefore\ \frac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + \cdots + x^n = 1 + S_n$

$\therefore\ S_n = -1 + \frac{1 - x^{n+1}}{1 - x}$, $-1 < x < 1$ for limiting sum

As $n \to \infty$, $S_n \to S \to -1 + \frac{1}{1 - x}$

As $x \to 1$, $S \to \infty$; as $x \to -1$, $S \to -\frac{1}{2}\ \therefore\ S \in \left(-\frac{1}{2}, \infty\right)$

Q31a $A = \frac{\theta}{2\pi} \times \pi\left((x + 1)^2 - 1^2\right) = \frac{\theta}{2}x(x + 2)$

$P = 2x + \frac{\theta}{2\pi} \times 2\pi((x + 1) + 1) = 2x + \theta(x + 2) = 2x + \frac{\theta x(x + 2)}{x} = 2x + \frac{2A}{x}$

Q31b Given A is constant, $\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 2 - \frac{\theta x(x + 2)}{x} = 0$

$\therefore\ \theta = \frac{2x}{x + 2} = 2 - \frac{4}{x + 2}$. As $x \to \infty$, $\theta \to 2$ from below $\therefore\ \theta < 2$

Please inform mathline@itute.com re conceptual and/or mathematical errors