2024 Specialist Mathematics Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	E	B	C	E	D	C	A	C	A	B	A	A	D	C	E	D	D	C	B

Question 1 Graph $y=-\frac{2}{x}$ to see the sections $x<-2$ or $x\ge 1$, i.e. $x\in R\setminus[-2,1)$ A

Question 2 E

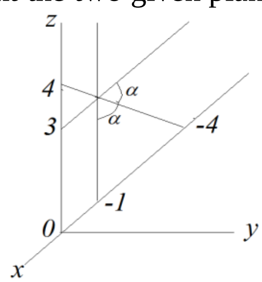
Question 3 $\frac{12\pi}{5a}-\frac{2\pi}{5a}=\frac{2\pi}{a}=1$ period, volume $=2\times\pi\int_0^{\frac{\pi}{a}}\sin^2(ax)\,dx=\pi\int_0^{\frac{\pi}{a}}(1-\cos(2ax))\,dx=\frac{\pi^2}{a}$ B

Question 4 $\int\frac{du}{u}=\log_e|u|+C$, $\int_{1-\sqrt{3}}^{1}\frac{2x-2}{x^2-2x-3}\,dx=\left[\log_e\left|x^2-2x-3\right|\right]_{1-\sqrt{3}}^{1}=\log_e 4$ C

Question 5 The plane $\frac{20}{60}x+\frac{15}{60}y+\frac{12}{60}z=1$ cuts the axes at $x=3$, $y=4$ and $z=5$. All four planes form a triangular pyramid, volume $=\frac{1}{3}Ah=\frac{1}{3}\times\left(\frac{1}{2}\times 3\times 4\right)\times 5=10$ E

Question 6 By checking $\frac{dy}{dx}$ at $x=0$ and different y values D

Question 7 The lines on the z-x plane represent the two given planes and the bisecting plane $z-x=4$. C



Question 8 $\frac{dv}{dx}=1+v^2$, $\int\frac{1}{1+v^2}\,dv=\int dx$, and $v=0$ at $x=0$ $\therefore$ $x=\tan^{-1}(v)$

At $x=\frac{\pi}{3}$, $\tan(x)=\sqrt{3}$ $\therefore$ $a=v\frac{dv}{dx}=v(1+v^2)=\tan(x)(1+\tan^2(x))=4\sqrt{3}$ A

Question 9 Since $z^6-1=(z-1)(z^5+z^4+z^3+z^2+z+1)=0$ and the six roots of $z^6-1=0$ space out equally on the circumference of a unit circle, they are: 1 from $z-1=0$

and $z=-1, cis\left(\pm\frac{\pi}{3}\right), cis\left(\pm\frac{2\pi}{3}\right)$ from $z^5+z^4+z^3+z^2+z+1=0$ $\therefore$ the sum of these five roots $=-1$ C

Question 10 $z^n=i=cis\left(\frac{\pi}{2}+2k\pi\right)=cis\left(\frac{(4k+1)\pi}{2}\right)$ where $k\in Q$ $\therefore$ $z=cis\left(\frac{(4k+1)\pi}{2n}\right)$

Comparing: $cis\left(\frac{(4k+1)\pi}{2n}\right)=cis\left(\frac{5\pi}{14}\right)$ $\therefore$ $n=7$ and $k=1$. When $k=\pm 2$, $z=cis\left(\frac{9\pi}{14}\right)$ and $-i$ respectively. A

Question 11 $\frac{dy}{dx}=x+y$, $x_0=1$, $y_0=1$

$y_1\approx y_0+h\frac{dy}{dx}=1+0.1(1+1)=1.2$, $y_2\approx y_1+h\frac{dy}{dx}=1.2+0.1(1.1+1.2)=1.43$ B

Question 12 $x = e^t - 2t$, $\frac{dx}{dt} = e^t - 2$; $y = 4\sqrt{2}\,e^{\frac{t}{2}}$, $\frac{dy}{dt} = 2\sqrt{2}\,e^{\frac{t}{2}}$

Distance $= \int_0^{\frac{1}{2}} \sqrt{\left(e^t - 2\right)^2 + \left(2\sqrt{2}\,e^{\frac{t}{2}}\right)^2}\,dt = \int_0^{\frac{1}{2}} \left(e^t + 2\right)dt = \sqrt{e}$ A

Question 13 Notice that $|\tilde{a}| = |\tilde{b}| = |\tilde{c}| = 2$, a vector making the same acute angle with each of the three given vectors is $\tilde{a} + \tilde{b} + \tilde{c} = \sqrt{2}\left(\tilde{i} + \tilde{j} + \tilde{k}\right)$.: a vector making the same obtuse angle is $-\sqrt{2}\left(\tilde{i} + \tilde{j} + \tilde{k}\right)$
.: a possible vector is in the direction of $-\left(\tilde{i} + \tilde{j} + \tilde{k}\right)$ A

Question 14 The plane $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$ has *x*-intercept of 1, *y*-intercept of 2 and *z*-intercept of 3.
Let $\tilde{a} = -\tilde{i} + 2\tilde{j} + 0\tilde{k}$ be a vector from the *x*-intercept to the *y*-intercept, $b = -\tilde{i} + 0\tilde{j} + 3\tilde{k}$ be a vector from the *x*-intercept to the *z*-intercept, and $\tilde{n} = \tilde{a} \times \tilde{b} = 6\tilde{i} + 3\tilde{j} + 2\tilde{k}$.: $\hat{n} = \frac{1}{7}\left(6\tilde{i} + 3\tilde{j} + 2\tilde{k}\right)$

$\cos\theta = \hat{p} \bullet \hat{n} = \frac{1}{\sqrt{14}}\left(\tilde{i} + 2\tilde{j} + 3\tilde{k}\right) \bullet \frac{1}{7}\left(6\tilde{i} + 3\tilde{j} + 2\tilde{k}\right) \approx 0.6872$.: $\theta \approx 46.6°$ D

Question 15 $\tilde{r}(t) = (\cos(t) + 1)\tilde{i} + (\sin(t) + 2)\tilde{j} + 3\tilde{k}$, $\dot{\tilde{r}}(t) = -\sin(t)\tilde{i} + \cos(t)\tilde{j}$, $\ddot{\tilde{r}}(t) = -\cos(t)\tilde{i} - \sin(t)\tilde{j}$
.: $\ddot{\tilde{r}}(t) = \tilde{r}_0 - \tilde{r}(t)$ C

Question 16 $\tilde{r}(t) = \tilde{r}_0 + \tilde{s}(t) = (5 + t)\tilde{i} + (1 + 4t)\tilde{j} + (2 - 2t)\tilde{k}$, At the plane $z = 0$.: $2 - 2t = 0$ i.e. when $t = 1$
.: $x = 6$ and $y = 5$.: $(6, 5, 0)$ E

Question 17 D

Question 18 $250a + 280b = 263.3$ and $36a^2 + 16b^2 = 14.27$.: $a \approx 0.5566$ and $b \approx 0.4434$.: $\frac{a}{b} \approx 1.25$ D

Question 19 $E\left(\bar{X}\right) \approx \mu = 12$ and $sd\left(\bar{X}\right) \approx \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{625}} = 0.2$, $\Pr\left(\bar{X} > 12.1\right) \approx 0.3$ C

Question 20 $\mu \approx \bar{x} = 11.982$ and $\sigma \approx \frac{s}{\sqrt{n}} = \frac{0.210}{\sqrt{625}} = 0.0084$

For 80% confidence interval, $z = invNorm(0.9) \approx 1.28155$
.: $(11.982 - 1.28 \times 0.0084,\ 11.982 + 1.28 \times 0.0084) \approx (11.971,\ 11.993)$ B

SECTION B Extended-answer questions

Question 1

a. Prove $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \le \frac{2n-1}{n}$, $n \ge 1$

For $n = 1$, $1 \le 1$ is true.

Assume for $n = k$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \le \frac{2k-1}{k}$

Consider $n = k + 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \le \frac{2k-1}{k} + \frac{1}{(k+1)^2}$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \le \frac{(2k-1)(k+1)^2 + k}{k(k+1)^2}$$

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \le \frac{k(2k+1)(k+1) - 1}{k(k+1)^2} = \frac{2k+1}{k+1} - \frac{1}{k(k+1)^2} \le \frac{2(k+1)-1}{(k+1)}$ is true for $n = k + 1$

.: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \le \frac{2n-1}{n}$ $\forall$ $n \ge 1$

b. $x \pm 1$ is divisible by $3 \Rightarrow x^6 - 1$ is divisible by 3

Proof: $x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x^3 - 1)(x + 1)(x^2 - x + 1)$ or $(x - 1)(x^2 + x + 1)(x^3 + 1)$

$\therefore$ $x \pm 1$ is divisible by $3 \Rightarrow x^6 - 1$ is divisible by 3

$x^6 - 1$ is divisible by $3 \Rightarrow$ $x \pm 1$ is divisible by 3

Proof: $x^6 - 1 = (x^3 - 1)(x^3 + 1)$ is divisible by $3 \Rightarrow x^3 \pm 1$ is divisible by 3

If $x^3 + 1 = (x + 1)(x^2 - x + 1)$ is divisible by 3, then $x + 1$ or $x^2 - x + 1$ is divisible by 3

If $x^2 - x + 1$ is divisible by 3, let $x^2 - x + 1 = 3n$ for some n (to be determined)

$x^2 - x + 1 - 3n = 0$, $x = \dfrac{1 \pm \sqrt{1 - 4(1 - 3n)}}{2} = \dfrac{1 \pm \sqrt{12m - 3}}{2} = \dfrac{1 \pm \sqrt{3(4m - 1)}}{2}$

Let $4m - 1 = 3p^2$, $p \in Q^+$, $x = \dfrac{1 \pm 3p}{2}$, $x + 1 = \dfrac{3 \pm 3p}{2} = 3\left(\dfrac{1 \pm p}{2}\right)$ $\therefore$ $x + 1$ is divisible by 3

Similarly, if $x^3 - 1 = (x - 1)(x^2 + x + 1)$ is divisible by 3, then $x - 1$ is divisible by 3

Question 2

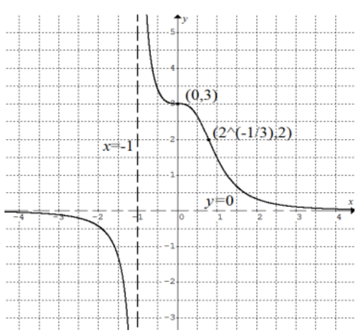
a. $\dfrac{3}{x^3 + 1} = \dfrac{3}{(x + 1)(x^2 - x + 1)} = \dfrac{A}{x + 1} + \dfrac{Bx + C}{x^2 - x + 1}$ $\therefore$ $A(x^2 - x + 1) + (Bx + C)(x + 1) = 3$

Comparing coefficients, $A = 1, B = -1$ and $C = 2$ $\therefore$ $y = \dfrac{1}{x + 1} - \dfrac{x - 2}{x^2 - x + 1}$

b. $y = \dfrac{3}{x^3 + 1}$, $\dfrac{dy}{dx} = \dfrac{-9x^2}{(x^3 + 1)^2} = 0$ $\therefore$ $x = 0$, $y = 3$, stationary point $(0, 3)$

$\dfrac{d^2y}{dx^2} = \dfrac{(x^3 + 1)^2(-18x) + 9x^2 2(x^3 + 1)3x^2}{(x^3 + 1)^4} = \dfrac{18x(2x^3 - 1)}{(x^3 + 1)^3} = 0$ $\therefore$ $x = 0, y = 3$ or $x = \dfrac{1}{\sqrt[3]{2}}, y = 2$, points of inflection are $(0, 3)$ and $\left(\dfrac{1}{\sqrt[3]{2}}, 2\right)$. y-intercept $(0, 3)$; asymptotes are $x = -1$ and $y = 0$.

c.



d. $a = 2, b = -1, c = -3, d = \dfrac{1}{2}, e = \dfrac{3}{4}$

e. Area

$$= \int_0^1 \frac{3}{x^3 + 1}\,dx = \int_0^1 \left(\frac{1}{x + 1} - \frac{1}{2}\left(\frac{2x - 1}{x^2 - x + 1} - \frac{3}{(x - \frac{1}{2})^2 + \frac{3}{4}}\right)\right)dx$$

$$= \left[\log_e(x + 1) - \frac{1}{2}\left(\log_e(x^2 - x + 1) - 2\sqrt{3}\tan^{-1}\left(\frac{2(x - \frac{1}{2})}{\sqrt{3}}\right)\right)\right]_0^1$$

$$= \log_e(2) + 2\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \log_e(2) + \frac{\pi}{\sqrt{3}}$$

Question 3

a. $\frac{dx}{dt}=xt$ and $x=1$ at $t=0$, $\int \frac{1}{x}\,dx=\int t\,dt$, $\log_e(x)=\frac{t^2}{2}$, $x=e^{\frac{t^2}{2}}$. At $t=2$, $x=e^2$.

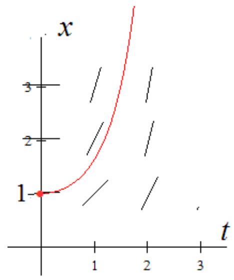
b. $v=\frac{dx}{dt}=xt=te^{\frac{t^2}{2}}$

c. $\log_e(x)=\frac{t^2}{2}$, $t=\sqrt{2\log_e(x)}$, $v=xt=x\sqrt{2\log_e(x)}$

d. $a=\frac{dv}{dt}=t\frac{dx}{dt}+x=t\left(te^{\frac{t^2}{2}}\right)+e^{\frac{t^2}{2}}=\left(t^2+1\right)e^{\frac{t^2}{2}}$

e. $a=\frac{dv}{dt}=t\frac{dx}{dt}+x=\sqrt{2\log_e(x)}\left(x\sqrt{2\log_e(x)}\right)+x=x\left(2\log_e x+1\right)$

f. g.



Question 4

a. $v\frac{dv}{dx}=a$, $\int_u^v v\,dv=\int_0^s a\,dx$, $\left[\frac{v^2}{2}\right]_u^v=\left[ax\right]_0^s$, $\frac{v^2}{2}-\frac{u^2}{2}=as$ $\therefore\ v^2=u^2+2as$

b. $\tilde{r}(t)=2\pi\cos\left(\frac{\pi t}{2}\right)\tilde{i}+2\pi\sin\left(\frac{\pi t}{2}\right)\tilde{j}+\pi t\tilde{k}$ at time $t\ge 0$,

$\tilde{v}(t)=\frac{d\tilde{r}(t)}{dt}=-\pi^2\sin\left(\frac{\pi t}{2}\right)\tilde{i}+\pi^2\cos\left(\frac{\pi t}{2}\right)\tilde{j}+\pi\tilde{k}$, $|\tilde{v}|^2=\tilde{v}\bullet\tilde{v}=\pi^4\sin^2\left(\frac{\pi t}{2}\right)+\pi^4\cos^2\left(\frac{\pi t}{2}\right)+\pi^2=\pi^4+\pi^2$

$\therefore$ speed $=|\tilde{v}|=\sqrt{\pi^4+\pi^2}=\pi\sqrt{\pi^2+1}$ is constant.

c. Time to complete one cycle $=$ one period $\frac{2\pi}{\frac{\pi}{2}}=4$

Distance travelled $L=\int_0^4\sqrt{\pi^4\sin^2\left(\frac{\pi t}{2}\right)+\pi^4\cos^2\left(\frac{\pi t}{2}\right)+\pi^2}\,dt=\int_0^4\sqrt{\pi^4+\pi^2}\,dt=\left[\pi\sqrt{\pi^2+1}\,t\right]_0^4=4\pi\sqrt{\pi^2+1}$

Constant speed $v=\frac{4\pi\sqrt{\pi^2+1}}{4}=\pi\sqrt{\pi^2+1}$

d. $\tilde{v}(t)=\frac{d\tilde{r}(t)}{dt}=-\pi^2\sin\left(\frac{\pi t}{2}\right)\tilde{i}+\pi^2\cos\left(\frac{\pi t}{2}\right)\tilde{j}+\pi\tilde{k}$, $\tilde{a}=\frac{d\tilde{v}(t)}{dt}=-\frac{\pi^3}{2}\cos\left(\frac{\pi t}{2}\right)\tilde{i}-\frac{\pi^3}{2}\sin\left(\frac{\pi t}{2}\right)\tilde{j}$

$\tilde{a}\bullet\tilde{v}=\frac{\pi^5}{2}\sin\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi t}{2}\right)-\frac{\pi^5}{2}\sin\left(\frac{\pi t}{2}\right)\cos\left(\frac{\pi t}{2}\right)=0$ $\therefore\ \tilde{a}(t)$ is perpendicular to $\tilde{v}(t)$.

Since $\tilde{a}$ is $\perp$ to $\tilde{v}$, the change in velocity is $\perp$ to $\tilde{v}$, not in the direction of $\tilde{v}$, hence the speed is constant.

e. $u=\pi\sqrt{\pi^2+1}$, $v=2u$, $s=4\pi\sqrt{\pi^2+1}$ $\therefore\ a=\frac{v^2-u^2}{2s}=\frac{3\pi^2\left(\pi^2+1\right)}{8\pi\sqrt{\pi^2+1}}=\frac{3}{8}\pi\sqrt{\pi^2+1}$

f. Distance travelled after two cycles $=2L=8\pi\sqrt{\pi^2+1}$, $v^2=\left(\pi\sqrt{\pi^2+1}\right)^2+2\left(\frac{3}{8}\pi\sqrt{\pi^2+1}\right)\left(8\pi\sqrt{\pi^2+1}\right)$

$\therefore$ speed $v=\pi\sqrt{7\left(\pi^2+1\right)}$; direction of motion is the same as $\tilde{v}(0)=\frac{1}{\sqrt{\pi^2+1}}\left(\pi\tilde{j}+\tilde{k}\right)$ $\therefore$ velocity $\tilde{v}=\sqrt{7}\pi\left(\pi\tilde{j}+\tilde{k}\right)$

Question 5

a. Total number of workers in the sample $=990$, $\bar{x}=\frac{125}{990}(35)+\frac{300}{990}(32)+\frac{230}{990}(32)+\frac{180}{990}(36)+\frac{155}{990}(34)\approx 33.42$

Variance $s^2=\left(\frac{125}{990}\right)^2(10)^2+\left(\frac{300}{990}\right)^2(8)^2+\left(\frac{230}{990}\right)^2(9)^2+\left(\frac{180}{990}\right)^2(10)^2+\left(\frac{155}{990}\right)^2(9)^2\approx 17.1344$, $s\approx 4.14$

b. $\Pr(33<\bar{X}<34)=normalcdf(33, 34, 33.42, 4.14)\approx 0.0961$

c. $\mu\approx 33.42$; $\sigma=\frac{s}{\sqrt{n}}=\frac{4.14}{\sqrt{990}}\approx 0.1316$

d. $\Pr(33<X<34)=normalcdf(33, 34, 33.42, 0.1316)\approx 0.9993$

e. For 90% confidence interval, $z=invNorm(0.95)\approx 1.6449$

.: interval is $(33.42-1.6449\times 0.1316, 33.42+1.6449\times 0.1316)\approx(33.20, 33.64)$

f. The mean travelling time is expected to increase

.: H_0 : there is no change, $\mu=33.42$, and H_1: $\mu>33.42$

g. $\bar{x}=33.72$, $s=4.25$.: $\mu\approx 33.72$ and $\sigma=\frac{4.25}{\sqrt{990}}\approx 0.1351$

95% confidence interval is $(33.72-1.96\times 0.1351, 33.72+1.96\times 0.1351)\approx(33.45, 33.98)$

Since 33.42 is below the interval .: road works increase travelling time

h. Type 1 error: Reject H_0 when H_0 is true, i.e. reject that there is no change in travelling time (or accept that there is an increase) when there is no change.

Type 2 error: Accept H_0 when H_0 is false, i.e. accept that there is no change in travelling time when there is an increase.

