2024 Specialist Mathematics

Year 12 Modelling Task

(Time allowed: 2.0 hours plus)

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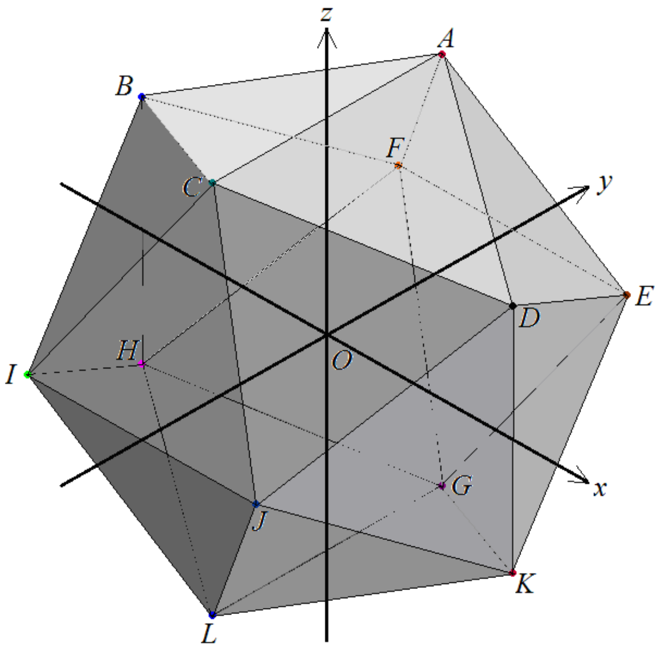
Modelling Task

Theme: Geometry of Regular Icosahedrons

Assumed knowledge: Co-ordinate geometry, position and free vectors, scalar and vector products, vector proofs, vector equations of lines and planes, geodesics, differential and integral calculus, arc length, surface area and volume, CAS

An **icosahedron** is a three-dimensional shape.
An icosahedron has **12 vertices**, **20 triangular faces** and **30 edges**. For a **regular** icosahedron, the faces are identical equilateral triangles, and the edges are straight line segments of the same length.
The following diagram shows a regular icosahedron.
Point O is the centre of the regular icosahedron.
Cartesian co-ordinate axes x, y and z are added to the diagram with point O as the origin of the axes.



The vertices are labelled as $A, B, C, \cdots, K$ and L in the above diagram.
Edges AD, DE and EA etc. have equal lengths.
All faces formed by any 3 adjacent vertices are equilateral triangles. For examples, faces ADE and CDJ are equilateral triangles.

The co-ordinates of the 12 vertices are:

$$A\left(0,\ 2,\ 1+\sqrt{5}\right),\ B\left(-1-\sqrt{5},\ 0,\ 2\right),\ C\left(0,\ -2,\ 1+\sqrt{5}\right),\ D\left(1+\sqrt{5},\ 0,\ 2\right),\ E\left(2,\ 1+\sqrt{5},\ 0\right),\ F\left(-2,\ 1+\sqrt{5},\ 0\right)$$

$$G\left(0, 2, -1-\sqrt{5}\right),\ H\left(-1-\sqrt{5}, 0, -2\right),\ I\left(-2, -1-\sqrt{5}, 0\right),\ J\left(2, -1-\sqrt{5}, 0\right),\ K\left(1+\sqrt{5}, 0, -2\right),\ L\left(0, -2, -1-\sqrt{5}\right)$$

Part I (80 minutes plus)

Let $\tilde{i}$, $\tilde{j}$ and $\tilde{k}$ be unit vectors in the direction of positive x, y and z axes respectively. Lengths are measured in arbitrary unit.

a. Select any three non-adjacent vertices.
In terms of $\tilde{i}$, $\tilde{j}$ and $\tilde{k}$ write down the position vectors of your chosen vertices.

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b. Calculate the magnitude of each position vector of your chosen vertices in part a.

c. Write down a hypothesis about the distance of any vertex of a regular icosahedron from centre O.

d. Consider the following statement:
There exists a **sphere** (called a circumsphere) with all vertices of the regular icosahedron lying on its surface.
Discuss the validity of the statement.

e. Select 3 adjacent vertices. Use vectors to show the face bounded by the vertices is an equilateral triangle.

f. Determine the equation of the plane of the triangular face in part e.

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g. Determine a vector equation and the parametric equations of any one of the three **line segments** forming the triangular face in part e.

h. The three medians of your selected triangle are concurrent, i.e. they intersect at the same point. This point of intersection divides each median in the ratio $2:1$.
Determine the position vector of this point. Hence write down the co-ordinates of this point.

i. Determine the Cartesian equation of the normal to the triangular face at the point of intersection of the medians.
Note: If you are unable to do part h, use the following information to complete part i and part j.
Consider equilateral ΔABC instead.
The point of intersection of the medians of ΔABC has co-ordinates $\left(\frac{-1-\sqrt{5}}{3},\ 0,\ \frac{4+2\sqrt{5}}{3}\right)$.

j. Show/explain that the normal passes through the centre of the icosahedron.

k. Write a general statement about the concurrency of the normals to the triangular faces of a regular icosahedron.

End of Part I

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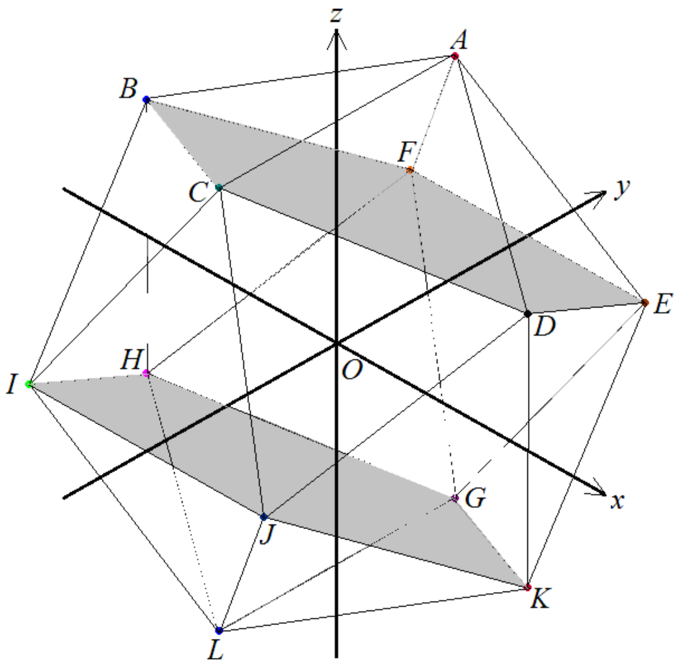
Part II (80 minutes plus)

An icosahedron has **12 vertices**, **20 triangular faces** and **30 edges**. For a **regular** icosahedron, the faces are identical equilateral triangles, and the edges are straight line segments of the same length.
The following diagram shows a regular icosahedron.
Point O is the centre of the regular icosahedron.
Cartesian co-ordinate axes x, y and z are added to the diagram with point O as the origin of the axes.



The vertices are labelled as $A, B, C, \cdots, K$ and L in the above diagram.
Edges AD, DE and EA etc. have equal lengths.
All faces formed by any 3 adjacent vertices are equilateral triangles. For examples, faces ADE and CDJ are equilateral triangles.

The co-ordinates of the 12 vertices are:

$$A\left(0,\ 2,\ 1+\sqrt{5}\right),\ B\left(-1-\sqrt{5},\ 0,\ 2\right),\ C\left(0,\ -2,\ 1+\sqrt{5}\right),\ D\left(1+\sqrt{5},\ 0,\ 2\right),\ E\left(2,\ 1+\sqrt{5},\ 0\right),\ F\left(-2,\ 1+\sqrt{5},\ 0\right)$$

$$G\left(0, 2, -1-\sqrt{5}\right),\ H\left(-1-\sqrt{5}, 0, -2\right),\ I\left(-2, -1-\sqrt{5}, 0\right),\ J\left(2, -1-\sqrt{5}, 0\right),\ K\left(1+\sqrt{5}, 0, -2\right),\ L\left(0, -2, -1-\sqrt{5}\right)$$

The point of intersection of the medians of ΔABC has co-ordinates $\left(\dfrac{-1-\sqrt{5}}{3},\ 0,\ \dfrac{4+2\sqrt{5}}{3}\right)$.

Volume of any 'pyramid' shape solid $= \dfrac{1}{3} \times \text{base area} \times \text{height}$ where the base can be any 2D shape

a. Select any three points out of B, C, D, E and F, find a Cartesian equation of the plane through your selected points. Show that the remaining two points also lie on the plane.

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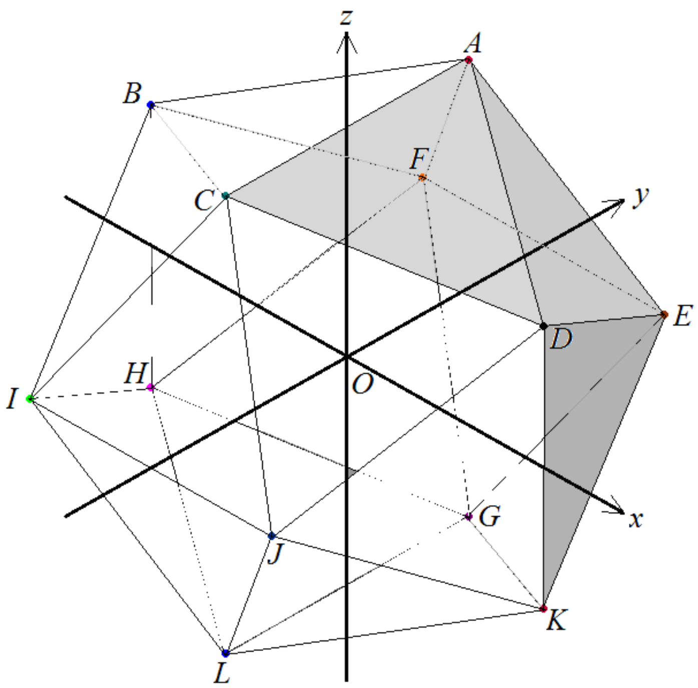
b. Show that plane $BCDEF$ and plane $GHIJK$ are parallel.



The co-ordinates of the 12 vertices are:

$A(0,\ 2,\ 1+\sqrt{5})$, $B(-1-\sqrt{5},\ 0,\ 2)$, $C(0,\ -2,\ 1+\sqrt{5})$, $D(1+\sqrt{5},\ 0,\ 2)$, $E(2,\ 1+\sqrt{5},\ 0)$, $F(-2,\ 1+\sqrt{5},\ 0)$

$G(0, 2, -1-\sqrt{5})$, $H(-1-\sqrt{5}, 0, -2)$, $I(-2, -1-\sqrt{5}, 0)$, $J(2, -1-\sqrt{5}, 0)$, $K(1+\sqrt{5}, 0, -2)$, $L(0, -2, -1-\sqrt{5})$

c. Find the angle (in degrees correct to 2 decimal places) between ΔADE and ΔKDE at the intersection DE.

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d. Find the angle (in degrees correct to 2 decimal places) between ΔADE and ΔADC at the intersection AD.

e. Write a general statement about the angle between any two adjacent faces.

f. Calculate the area of ΔABC. Hence determine the total surface area of the regular icosahedron.

g. Calculate the volume of the pyramid having ΔABC as its base and centre O as its vertex.
Hence calculate the percentage of the volume of the circumsphere is the volume of a regular icosahedron.

h. Calculate the shortest distance **on** the surface of the circumsphere from vertex A to vertex L.

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i. Find the area of the regular pentagon $BCDEF$.
Find the shortest distance from vertex A to the pentagon $BCDEF$.
Hence find the volume of the pyramid having pentagon $BCDEF$ as its base and point A as its vertex.

j. Find the volume bounded by the circumsphere and the plane $BCDEF$. (Minor section)

k. Find the surface area of the shape bounded by the circumsphere and the plane $BCDEF$. (Minor section)

End of Part II

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