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2024 Specialist Mathematics

Year 12 Problem Solving Task

(Time allowed: 2.0 hours plus)

Problem Solving Task

Theme: Continuous random variables and probability distributions

Assumed knowledge: Functions and graphs, algebra, probability density functions, normal distributions and approximation, transformations, calculus, use of CAS

Part I (70 minutes plus)

Random variable X has a normal distribution with probability density function

$$f(x) = Ae^{-(0.0345x-2.2420)^2}$$, correct to 4 decimal places.

a. Use calculus to show that $A \approx 0.0195$.

b. By differentiation show that the mean of X is $\mu_X \approx 64.9855$.

c. Use CAS to evaluate $\int_{-\infty}^{\infty} xf(x)dx$ to 4 decimal places. Comment.

d. Use CAS to evaluate $\int_{-\infty}^{\infty} x^2 f(x)dx$. Hence show that the standard deviation of X is $\sigma_X \approx 20.4958$.

Random variable Z has probability density function $f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$, the standard normal distribution.

e. Describe a sequence of transformations required to change $f(x)$ to $f(z)$.

Random variable Y has probability density function $g(y)=Be^{-(0.0245y-2.2420)(0.0345y-2.5450)}$, correct to 4 decimal places.

f. Use calculus to show $B \approx 0.0153$

g. Show that the mean and standard deviation of Y are $\mu_Y \approx 82.6392$ and $\sigma_Y \approx 24.3216$ respectively.

h. Investigate whether the probability distribution of random variable Y is normal or not.

In a population sometime ago, the weight distribution of females was given by the distribution of X, $f(x)=0.0195e^{-(0.0345x-2.2420)^2}$, and the weight distribution of males was given by the distribution of Y, $g(y)=0.0153e^{-(0.0245y-2.2420)(0.0345y-2.5450)}$. Measure weights in kg

The ratio *number of females* : *number of males* is $\alpha:\beta$ where $\alpha+\beta=1$.
Weight of an individual in the population was random variable W.

i. Express W in terms of X, Y and α.

j. Given that $W=0.54X+0.46Y$, show that $\mu_W \approx 73.1062$ and $\sigma_W \approx 15.7374$.

k. Prove that $\forall c \in R$, $\Pr(Y<c)-\Pr(X>c)=\Pr(X<c)-\Pr(Y>c)$.

l. Find the value(s) of $c \in R$ such that $f(c)=g(c)$, correct to 3 decimal places.

m. Assume that the distributions of X and Y in the population remain the same, discuss the effect on the mean weight of the population when the value of the ratio $\dfrac{\alpha}{\beta}$ was increased.
Illustrate your answer with **two** examples.

End of Part I

Part II (70 minutes plus)

From **Part I**:

In a population sometime ago, the weight distribution of females was given by the distribution of X, $f(x) = 0.0195e^{-(0.0345x-2.2420)^2}$, and the weight distribution of males was given by the distribution of Y, $g(y) = 0.0153e^{-(0.0245y-2.2420)(0.0345y-2.5450)}$. Measure weights in kg

$\mu_X \approx 64.9855$ and $\sigma_X \approx 20.4958$, $\mu_Y \approx 82.6392$ and $\sigma_Y \approx 24.3216$

Weight of an individual in the population was random variable W where $W = 0.54X + 0.46Y$, $\mu_W \approx 73.1062$ and $\sigma_W \approx 15.7374$.

Random samples of size 20 were taken from the population. Let random variable $\overline{W}$ be the sample mean and its expectation is $E(\overline{W}) = \mu_{\overline{W}}$.

a. Find the value of δ such that $\Pr(\mu_{\overline{W}} - \delta < \overline{W} < \mu_{\overline{W}} + \delta) \approx 0.95$, correct to 2 decimal places.

b. Express δ in terms of σ_W, correct to 2 decimal places.

c. A random sample was taken: 13.7, 70.3, 68.7, 36.7, 55.2, 80.2, 73.1, 25.5, 49.3, 38.8, 55.4, 63.1, 81.0, 99.3, 29.6, 48.5, 68.1, 70.7, 80.3, 109.8
Determine the sample mean $\overline{w}$. Was this sample mean in the interval $(\mu_{\overline{W}} - \delta, \mu_{\overline{W}} + \delta)$ found in part a?

d. Discuss the effect of increasing the size of samples on the interval $(\mu_{\overline{W}} - \delta, \mu_{\overline{W}} + \delta)$.
Illustrate your answer with two examples.

e. What would be the minimum sample size for $\overline{w} \in (66, 80)$ such that $\Pr(66 < \overline{W} < 80) > 0.95$?

At present the ratio *number of females* : *number of males*, i.e. $\alpha : \beta$ has changed, where $\alpha + \beta = 1$.
The value of $\frac{\alpha}{\beta}$ now is different from its value sometime ago.

The weight distribution of females given by $f(x) = 0.0195e^{-(0.0345x-2.2420)^2}$, and the weight distribution of males given by $g(y) = 0.0153e^{-(0.0245y-2.2420)(0.0345y-2.5450)}$ remain the same.
Hence $\mu_X \approx 64.9855$ and $\sigma_X \approx 20.4958$, $\mu_Y \approx 82.6392$ and $\sigma_Y \approx 24.3216$.
The population size stays the same as the size sometime ago.

The following are values of weight w in a random sample of size 36 taken from the present population.

49.5	90.4	77.7	75.5	100.6	81.5	61.0	59.8	57.9
60.0	70.1	93.1	107.5	51.6	60.5	81.7	83.2	90.1
112.6	45.2	65.1	80.3	71.7	82.1	92.3	114.2	80.2
50.3	73.2	24.2	50.3	74.5	80.9	102.3	90.2	79.5

The sample is used to test whether there is significant change to the population mean due to the change in the ratio *number of females* : *number of males*, i.e. $\alpha : \beta$.

f. Use CAS to find the sample mean $\overline{w}$ and the sample standard deviation σ_w, correct to 4 decimal places.

g. Calculate the *p*-value to 4 decimal places, and use it to determine whether there is significant change to the population mean. Write a statement to summarise your conclusion, stating the chosen significant level.

h. Determine an approximate confidence interval, correct to 4 decimal places, for the population mean at the same significance level as in part g.

i. Use the sample mean as a point estimate of the population mean, estimate the value of the ratio $\frac{\alpha}{\beta}$, correct to 4 decimal places.

j. Use the approximate 95% confidence interval to estimate the range of values of α, correct to 3 decimal places.

End of Part II

