itute.com

2024 VCAA Specialist Mathematics Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
C	A	C	A	C	D	C	C	A	A

11	12	13	14	15	16	17	18	19	20
B	C	B	B	A	A	B	D	A	B

Q1 C

Q2 A

Q3 C

Q4 $\sin(x)=a$, $2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)=a$, $4\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=a^2$

$4\left(1-\cos^2\left(\frac{x}{2}\right)\right)\cos^2\left(\frac{x}{2}\right)=a^2$, $a^2-4\cos^2\left(\frac{x}{2}\right)+4\cos^4\left(\frac{x}{2}\right)=0$

$\cos^2\left(\frac{x}{2}\right)=\frac{1\pm\sqrt{1-a^2}}{2}$, $\cos\left(\frac{x}{2}\right)=-\frac{\sqrt{1\pm\sqrt{1-a^2}}}{\sqrt{2}}$ since

$\frac{3\pi}{4}<\frac{x}{2}<\pi$ A

Q5 C

Q6 $(3+ki)^2+4i(3+ki)+3=\left(-k^2-4k+12\right)+6(k+2)i$

Let $-k^2-4k+12=0$ and $6(k+2)\neq 0$ ∴ $k=2$ or -6 D

Q7 Simplify to $\frac{dy}{dx}=\frac{e^x}{e^y}2\sin x\sin y$. C

Q8 $y_0=1$, $y_1=1$, $y_2=1+h^2$,

$y_3=1+h^2+2h^2\left(1+h^2\right)^2=1.126528$ ∴ $h\approx 0.20$ C

Q9 $\int_0^{2\pi}\sqrt{(\sin t)^2+(1-\cos t)^2}=\int_0^{2\pi}2\sin\left(\frac{t}{2}\right)dt$ A

Q10

$\int_0^k 2\pi x\sqrt{5^2+12^2}\,dt=\int_0^k 10\pi t\sqrt{5^2+12^2}\,dt=\left[65\pi t^2\right]_0^k=65\pi k^2$ A

Q11 $\int_0^{100}(-0.6t+40)dt+\left(\frac{1}{2}\times 50\times {}^{-}20\right)=500$ B

Q12 $\frac{dx}{dt}=(k-1)e^{(k-1)t}$, $\frac{d^2x}{dt^2}=(k-1)^2e^{(k-1)t}=(k-1)^2x$

When $x=k+1$, $\frac{d^2x}{dt^2}=(k-1)^2(k+1)=\left(k^2-1\right)(k-1)$ C

Q13 Let $\hat{p}=\frac{1}{3}\left(2\tilde{i}-\tilde{j}+2\tilde{k}\right)$ and $\hat{q}=\frac{1}{\sqrt{40+m^2}}\left(2\tilde{i}+m\tilde{j}+6\tilde{k}\right)$

$\cos\theta=\hat{p}.\hat{q}=\frac{(16-m)^2}{9\left(40+m^2\right)}=\frac{13}{21}$ and $m>0$ ∴ $m=3$ B

Q14 $\tilde{s}=2\tilde{i}-2\tilde{j}+\tilde{k}$, scalar resolute of $\tilde{r}$ in the direction of $\tilde{s}$ $=-\sqrt{(-4)^2+4^2+(-2)^2}=-6$. Angle θ between $\tilde{r}$ and $\tilde{s}$ is given by $\cos\theta=\frac{-6}{9}=\frac{-2}{3}$ ∴ scalar resolute of $\tilde{s}$ in the direction of $\tilde{r}$ is $|\tilde{s}|\cos\theta=3\times\frac{-2}{3}=-2$. B

Q15 Parabolic path $y=1-2x^2$. Determine the times at $(0,1)$, $(1,-1)$ and $(-1,-1)$ to find the time 2π for the round trip. A

Q16 $\dot{r}_1=-\sin t\tilde{i}+\cos t\tilde{j}+\frac{\cos 2t}{\sqrt{\sin 2t}}\tilde{k}$, $\dot{r}_2=\cos t\tilde{i}-\sin t\tilde{j}+\frac{\cos 2t}{\sqrt{\sin 2t}}\tilde{k}$

$\dot{r}_1.\dot{r}_2=0$ ∴ $\sin 2t=\frac{1}{\sqrt{3}}$, only one solution in $\left(0,\frac{\pi}{2}\right)$ A

Q17 Unit vector in the direction of $\tilde{i}+\tilde{j}+\tilde{k}$ is $\hat{u}=\frac{1}{\sqrt{3}}\left(\tilde{i}+\tilde{j}+\tilde{k}\right)$

Vector resolute of $\tilde{r}_1(0)-\tilde{r}_2(0)=3\tilde{i}+2\tilde{j}-2\tilde{k}$ in the direction of $\hat{u}$ is $\left(\tilde{r}_1(0)-\tilde{r}_2(0)\right).\hat{u}\,\hat{u}=\tilde{i}+\tilde{j}+\tilde{k}$ ∴ vector resolute perpendicular to $\hat{u}$ is $\tilde{r}_1(0)-\tilde{r}_2(0)-\left(\tilde{i}+\tilde{j}+\tilde{k}\right)=2\tilde{i}+\tilde{j}-3\tilde{k}$

The shortest distance $=\left|2\tilde{i}+\tilde{j}-3\tilde{k}\right|=\sqrt{14}$ B

Q18 $\tilde{r}=(1-2t)\tilde{i}+(1+t)\tilde{j}+(3t-2)\tilde{k}$

∴ $3(1-2t)-2(1+t)+4(3t-2)=5$ ∴ $t=3$; $x=-5, y=4, z=7$ D

Q19 A

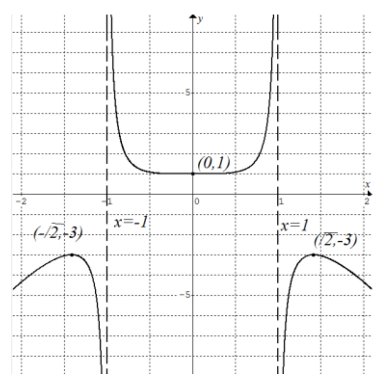
Q20 For a sample of 4 avocados, the sample mean $=4\times 200=800$ and sample standard deviation $=\sqrt{4\times 7.5^2}=15$

$F_4=0.7M_4>570$ ∴ $M_4>814.2857143$

$\Pr\left(M_4>814.2857143\right)\approx 0.1705$ B

SECTION B

Q1a



Q1bi $\frac{x^4-x^2+1}{1-x^2}=y$, $x^4+(y-1)x^2-(y-1)=0$

∴ $x^2=\frac{1-y+\sqrt{y^2+2y-3}}{2}$, $V=\int_1^6\frac{\pi}{2}\left(1-y+\sqrt{y^2+2y-3}\right)dy$

Q1bii $V\approx 11.2$

Q1c $b=-1$, $\frac{x^4-1}{1-x^2}=\frac{-\left(1-x^2\right)\left(x^2+1\right)}{\left(1-x^2\right)}=-\left(x^2+1\right)$ ∴ no asymptotes

Q1di Stationary points: $-2x\left(\left(x^2-1\right)^2-(b+1)\right)=0$

Exactly one: $\left(x^2-1\right)^2-(b+1)=x^4-2x^2-b\neq 0$, $\Delta<0$ ∴ $b<-1$

Q1dii Exactly three: $x^2=1-\sqrt{1+b}\leq 0$, $\sqrt{1+b}\geq 1$ ∴ $b\geq 0$

Q1diii Exactly five: $x^2=1-\sqrt{1+b}>0$ ∴ $b<0$ ∴ $-1<b<0$

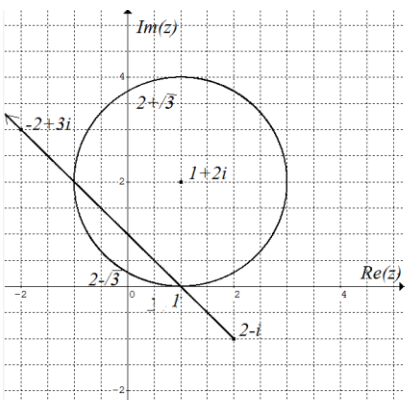
Q2a $\left|(x-1)+(y-2)i\right| = \left|(x-4)+yi\right|$

$(x-1)^2+(y-2)^2=(x-4)^2+y^2$ $\therefore$ $y=\frac{3}{2}x-\frac{11}{4}$

Q2b $\left|4-(1+2i)\right| = \left|3-2i\right| = \sqrt{13}$, radius $r=\frac{\sqrt{13}}{2}$

$z_c = \frac{4+(1+2i)}{2} = \frac{5}{2}+i$ $\therefore$ circle is $\left|z-\left(\frac{5}{2}+i\right)\right| = \frac{\sqrt{13}}{2}$

Q2c



Q2di See sketch in Q2c

Q2dii Ray: $\operatorname{Arg}(z-(2-i)) = \frac{3\pi}{4}$

Q2e Area of minor segment $=\frac{1}{4}\pi\times 2^2 - \frac{1}{2}\times 2\times 2 = \pi - 2$

Q3a Let $\frac{d}{dt}\left(\frac{dV}{dt}\right) = \frac{(240+5t^4)8-8t(20t^3)}{(240+5t^4)^2}=0$, $t\ge 0$ $\therefore$ $t=2$

Q3b $V=0.001\pi r^2$, $\frac{dr}{dt}=\frac{\frac{dV}{dt}}{\frac{dV}{dr}}=\frac{\frac{8t}{240+5t^4}}{0.002\pi r}$

Given $t=4$ and $r=6.54$, $\frac{dr}{dt}\approx 0.51$ m per day

Q3ci $u=\sqrt{5}\,t^2$, $\int \frac{8t}{240+5t^4}\,dt = \int \frac{\frac{4}{\sqrt{5}}}{240+u^2}\,du$

Q3cii $V=\frac{1}{5\sqrt{3}}\int \frac{\sqrt{240}}{240+u^2}du = \frac{1}{5\sqrt{3}}\arctan\left(\frac{t^2}{4\sqrt{3}}\right)$, $V=0$, $t=0$

Q3d As $t\to\infty$, $V\to \frac{1}{5\sqrt{3}}\times\frac{\pi}{2}=\frac{\pi}{10\sqrt{3}}$

$S.A.=\frac{V}{\text{depth}} \to \frac{\pi}{0.001\times 10\sqrt{3}} \approx 181.38$ m^2

Q3e

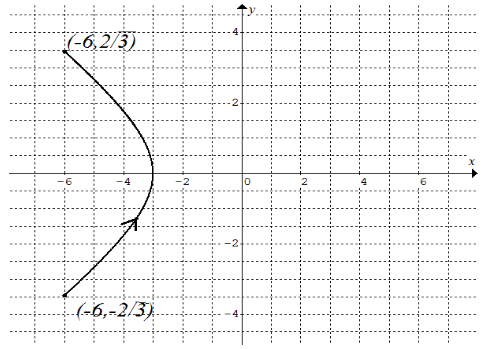
$V=\int_0^{T+5}\frac{8t}{240+5t^4}\,dt - 0.05T = \left[\frac{1}{5\sqrt{3}}\arctan\frac{t^2}{4\sqrt{3}}\right]_0^{T+5} - 0.05T = 0$

$\therefore$ $T\approx 10.7$ days from the start of clean up.

Q4a $\frac{x}{3}=\sec(t)$, $\frac{y}{2}=\tan(t)$; $\sec^2(t)-\tan^2(t)=1$, $\frac{2\pi}{3}\le t\le\frac{4\pi}{3}$

$\therefore$ $\frac{x^2}{9}-\frac{y^2}{4}=1$, $-6\le x\le -3$ and $-2\sqrt{3}\le y\le 2\sqrt{3}$

Q4b



Q4ci $\dot{r}_Y = 3\sec(t)\tan(t)\tilde{i} + 2\sec^2(t)\tilde{j}$

Speed $=\sqrt{9\sec^2(t)\tan^2(t)+4\sec^4(t)}$

$=\sqrt{9\sec^2(t)(\sec^2(t)-1)+4\sec^4(t)} = \sqrt{13\sec^4(t)-9\sec^2(t)}$

Q4cii $t=\pi$

Q4ciii 4 m per min.

Q4civ $(-3,0)$

Q4di $\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\sqrt{13\sec^4(t)-9\sec^2(t)}\,dt$

Q4dii Distance ≈ 9.4 m

Q4e $\tilde{r}_D-\tilde{r}_Y = (2-3t-3\sec(t))\tilde{i} + (4t-1-2\tan(t))\tilde{j} + (6-t)\tilde{k}$

$\left|\tilde{r}_D-\tilde{r}_Y\right| = \sqrt{(2-3t-3\sec(t))^2+(4t-1-2\tan(t))^2+(6-t)^2}$

Shortest distance ≈ 11.1 m

Q5a $\tilde{r}(\lambda)=\tilde{r}_A+\lambda\overrightarrow{AB} = \tilde{i}-2\tilde{j}+3\tilde{k}+\lambda(\tilde{i}-3\tilde{j}-4\tilde{k})$, $\lambda\in R$

Q5b $\tilde{r}_A=\tilde{i}-2\tilde{j}+3\tilde{k}$ and $\tilde{r}_1=(2-t)\tilde{i}+(1+2t)\tilde{j}+(t-3)\tilde{k}$

$\tilde{r}_1-\tilde{r}_A=(1-t)\tilde{i}+(2t+3)\tilde{j}+(t-6)\tilde{k}$,

$\left|\tilde{r}_1-\tilde{r}_A\right|=\sqrt{(1-t)^2+(2t+3)^2+(t-6)^2}$. Let $\frac{d}{dt}\left|\tilde{r}_1-\tilde{r}_A\right|=0$ $\therefore$ $t=-\frac{1}{4}$

Shortest distance $=\frac{5\sqrt{30}}{4}$ when $t=-\frac{1}{4}$

Q5c Let the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\therefore$ $\frac{1}{a}+\frac{-2}{b}+\frac{3}{c}=1$, $\frac{2}{a}+\frac{-5}{b}+\frac{-1}{c}=1$ and $\frac{0}{a}+\frac{2}{b}+\frac{-5}{c}=1$

Solve to obtain $a=\frac{19}{40}$, $b=\frac{19}{12}$ and $c=19$

Equation of the plane is $\frac{40x}{19}+\frac{12y}{19}+\frac{z}{19}=1$ or $40x+12y+z=19$.

Q5di $\frac{x}{6}+\frac{y}{-4}+\frac{z}{3}=1$

Axis intercepts are $X(6,0,0)$, $Y(0,-4,0)$ and $Z(0,0,3)$

Q5dii $\overrightarrow{XY}=-6\tilde{i}-4\tilde{j}$, $\overrightarrow{XZ}=-6\tilde{i}+3\tilde{k}$

Area $=\frac{1}{2}\left|\overrightarrow{XY}\times\overrightarrow{XZ}\right| = \frac{1}{2}\left|-12\tilde{i}+18\tilde{j}-24\tilde{k}\right| = \sqrt{261}=3\sqrt{29}$

Q6a $H_0:\ \mu$ is not less than 1000 ; $H_1:\ \mu$ is less than 1000

Q6bi p - value $= \Pr\left(z < \dfrac{997.5 - 1000}{\frac{4.2}{\sqrt{9}}}\right) \approx 0.037$

Q6bii Since $0.037 < 0.05$, the machine should be paused and adjusted because the mean volume is significantly less than 1000 ml at the 5% level of significance.

Q6c $\Pr\left(z < \dfrac{x - 1000}{\frac{4.2}{\sqrt{9}}}\right) > 5$.: $x > 997.6972$

Probability of a type II error $= \Pr(X > 997.6972 \mid \mu = 997) \approx 0.31$

Q6d $a \approx 996.7$, $b \approx 1003.3$

Q6e 95% confidence interval

$\left(1005 - 1.96 \times \dfrac{4}{\sqrt{50}},\ 1005 + 1.96 \times \dfrac{4}{\sqrt{50}}\right)$ simplify to

$(1003.9, 1006.1)$

Q6f $40 \times 0.95 = 38$

Q6g $1.96 \times \dfrac{4}{\sqrt{n}} \le 1$, $n \ge 61.4656$.: minimum $n = 62$

Please inform mathline@itute.com re conceptual and/or mathematical errors

