itute.com

Online & home tutors Registered business name: itute ABN: 96 297 924 083

2025 Mathematical Methods

Year 12

Application Task

Time allowed: 4 hours plus

Application Task

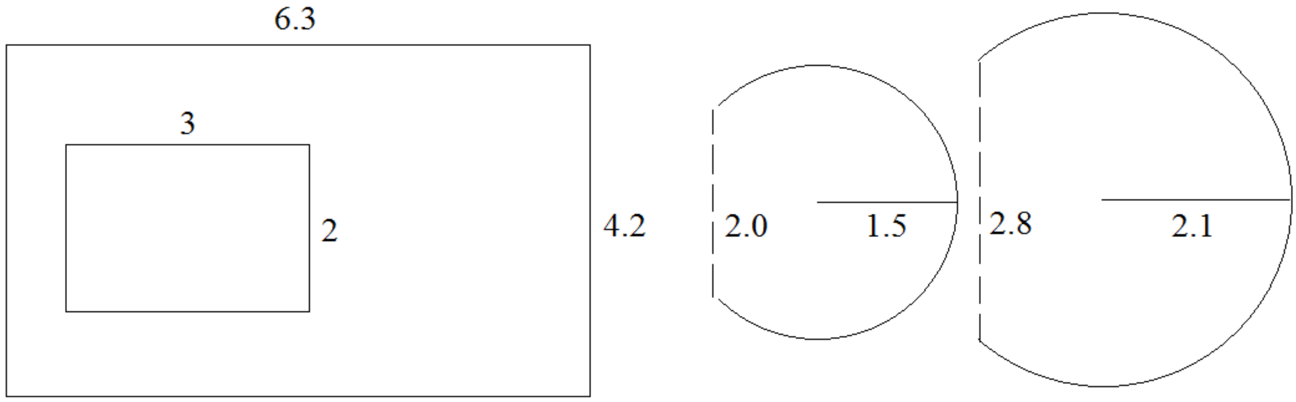
Theme: Parabolas and similarity

Assumed knowledge:

Functions and relations, algebra, quadratic functions, graphs, co-ordinate geometry, parameters, transformations, similar figures, differentiation and integration, tangent and normal to a curve, and CAS

Introduction:
In two dimensions two shapes (or curves) are similar when one is the dilation of the other in both dimensions by the same factor.

The following two rectangles are similar because one is the dilation of the other horizontally and vertically by the same factor. The two major arcs of circles are similar because the dilation factors in both directions are the same.

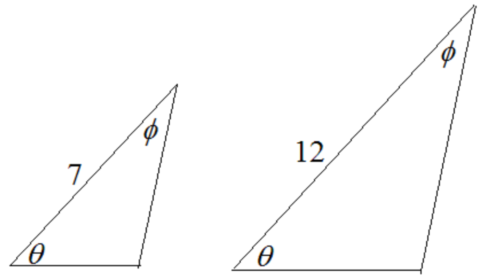


Task: Investigate similarity in parabolas.

Part I (80 minutes plus)

a. Explain why the two rectangles are similar, and the two arcs are similar. Compare and comment on the areas of the two rectangular regions, and the areas of the two major segments of the circles.

b. Show that the following two triangles are similar by finding the horizontal and vertical dilation factors.



c. Consider the parabolas with equations $y = x^2$ and $y = \frac{x^2}{4}$.

Sketch the two curves and the line $y = x$ on the same axes. Label the points of intersections.

d. Show that the parabolas with equations $y = x^2$ and $y = \frac{x^2}{4}$ are similar by using the intersections to find the horizontal and vertical dilation factors.

e. Consider the parabolas with equations $y = x^2$ and $y = \frac{x^2}{m}$.

Select five suitable and representative values of $m \neq 1, 4$ and $m \in \left(\frac{1}{8}, 8\right)$.

Sketch on the same set of axes the graphs of $y = x^2$ and $y = \frac{x^2}{m}$ for your five selected values of m. Label each graph with its m value.

f. Use the procedures in parts **c** and **d** to investigate whether the parabolas are similar. Comment on your findings.

g. The line $y = x$ intersects each of your parabolas at $(0,0)$ and one other point. Investigate the gradients of the tangents at the intersecting points other than $(0,0)$. Comment on your findings.

h. For $n \in R^+$, find the gradients of the tangents to the parabola $y = \frac{\left(n^2 + \frac{1}{2}\right)}{\left(n^2 + \sqrt{n}\right)} x^2$ at the intersections other than $(0,0)$ with the lines $y = x$ and $y = \frac{3x}{2}$.

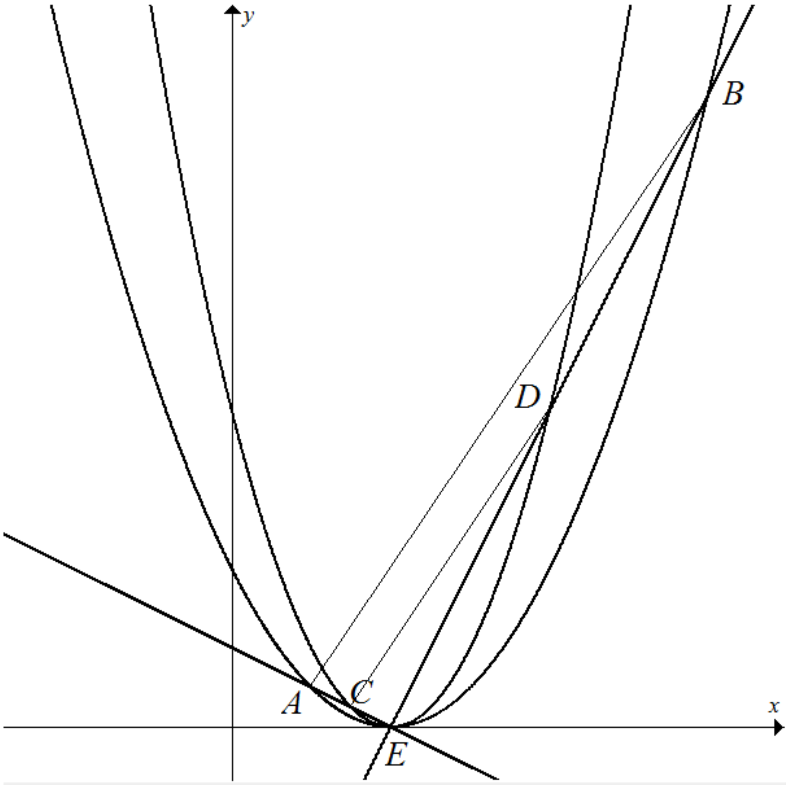
i. Show algebraically that $y = kx^2$ is the dilation of $y = x^2$ from both axes by the same factor $\frac{1}{k}$. Hence explain why all parabolas $y = ax^2 + bx + c$ are similar, where $a, b, c \in R$ and $a \neq 0$.

j. $y = ax^2$ is dilated from the x-axis by a factor of p, and from the y-axis by a factor of q, where $p, q \in R \setminus \{0\}$. State the equation of the transformed parabola.
The transformation of $y = ax^2$ can be considered as dilations by the same factor γ in both directions. Find the value of γ and in terms of p and q.

End of Part I

Part II (80 minutes plus)

Consider parabolas $y=(x-2)^2$ and $y=\frac{1}{2}(x-2)^2$, and lines $y=-\frac{1}{2}(x-2)$ and $y=2(x-2)$. Their graphs are shown below. A, B, C, D and E are points of intersection of the lines and the parabolas.



a. Label the intersections with their coordinates, the lines and the parabolas with their equations. Calculate the areas of ΔABE and ΔCDE.

b. $y=\frac{1}{2}(x-2)^2$ is the image of the $y=(x-2)^2$ under dilations from both axes by the same factor α. Determine the value of α. Express the ratio $\dfrac{\text{area of } \Delta ABE}{\text{area of } \Delta CDE}$ in terms of α.

c. Write down two definite integrals I_1 and I_2, where I_1 gives the area of the region between Line *EB* and parabola $y=\frac{1}{2}(x-2)^2$, and I_2 the area of the region between Line *ED* and parabola $y=(x-2)^2$. Evaluate I_1 and I_2. Express $\dfrac{I_1}{I_2}$ in terms of α.

Write a statement to hypothesise areas of similar regions bounded by parabolas and the same line in relation to dilation factor.

d. State a limitation to your hypothesis in part c.

e. Tangents to $y=\frac{1}{2}(x-2)^2$ at A and B makes acute angle θ. Tangents to $y=(x-2)^2$ at C and D makes acute angle ϕ.
By calculating θ and ϕ show that $\theta=\phi$.

Consider parabolas $y_1=hx^2$ and $y_2=kx^2$.

f. Parabolas y_1 and y_2 are similar. y_2 is the dilation of y_1 from both axes by the same factor λ. Determine the dilation factor λ in terms of h and k.

The line $y=mx$ where $m\in R^+$ cuts $y=hx^2$ at points O (origin) and W, where $h\in R^+$.
The tangent at point W makes angle θ with the x-axis.
The line $y=mx$ makes angle ϕ with the x-axis.

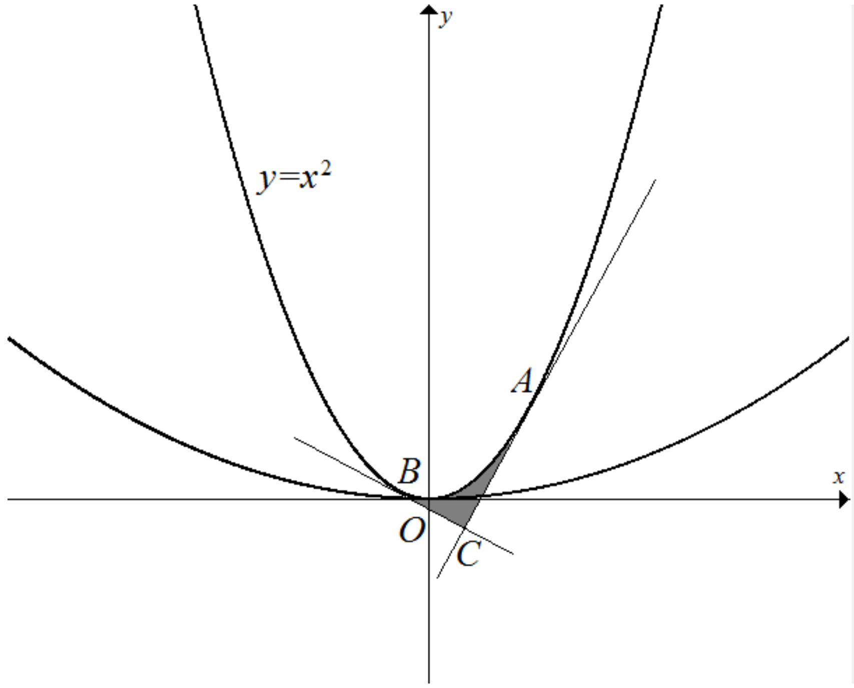
g. Investigate the effect of changing the value of h on angle θ. Comment on your findings.
(Hint: Choose an appropriate value of m and vary the value of h systematically)

h. Given $\tan(\theta-\phi)=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$, $\theta>\phi$.
Investigate the effect of changing the m values on $\tan(\theta-\phi)$ where $(\theta-\phi)$ is the angle between the line $y=mx$ and the tangent to the parabola $y=hx^2$ at point W. Comment
(Hint: Choose a suitable value of h and vary the value of m systematically)

i. Express $\tan(\theta-\phi)$ in terms of m. Use this expression to verify quantitatively your comment in part **h**.

End of Part II

Part III (80 minutes plus)



$A(a, a^2)$ and $B(b, b^2)$ are points on parabola $y = x^2$, $a > 0$.
The two tangents at A and B are perpendicular.
Another parabola is also shown above for part f.
The shaded region bounded by $y = x^2$ and the tangents is for part g.

a. Show that the equation of the tangent to parabola $y = x^2$ at $A(a, a^2)$ is $y = 2ax - a^2$.

b. The tangent to parabola $y = x^2$ at $B(b, b^2)$ is perpendicular to $y = 2ax - a^2$.
Show that $b = -\dfrac{1}{4a}$.

c. Show that the equation of the tangent to parabola $y = x^2$ at $B(b, b^2)$ perpendicular to $y = 2ax - a^2$ is $y = -\dfrac{x}{2a} - \dfrac{1}{16a^2}$.

d. Show that the above two tangents to parabola $y = x^2$ intersect at point $C\left(\dfrac{4a^2 - 1}{8a}, -\dfrac{1}{4}\right)$.

e. Find the lengths of AC and BC, hence show that the area of ΔABC is $\dfrac{1}{256}\left(\dfrac{1}{a} + 4a\right)^3$.

f. Does such a right-angle triangle exist for other parabolas of the form $y = kx^2$?
Verify your answer by two suitable values of $k \in R^+ \setminus \{1\}$ of your choice.

g. For parabola $y = x^2$, determine the value of a when the shaded region has a minimum area.

h. Show that ΔABC has minimum area when the value of a is the same as that found in part g.

i. Let the minimum area of the shaded region be Ω.
In terms of Ω, find the minimum area of the region bounded by parabolas $y = \frac{5}{2}x^2$ and its two perpendicular tangents.

j. A parabola is defined by $y = \frac{x^2}{4\alpha^2}$, where $\alpha > 0$.
A normal to the parabola at point M with $x = -\beta$, where $\beta > 0$, cuts the other side of the parabola at point N. The region bounded by the normal and the parabola has area Φ.
Sketch the parabola and the normal to illustrate the given information.
By similarity or otherwise, find the value of β, in terms of α, such that Φ is a minimum value.

End of Part III
End of Application Task