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2025 Mathematical Methods

Year 12

Modeling Task

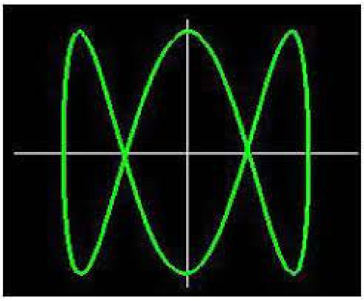
Time allowed: 2 hours plus

Modeling Task

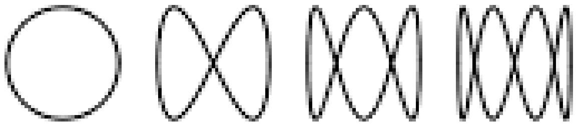
Theme: The ABC logo and Lissajous Curves

Introduction:

The Australian Broadcasting Corporation has the following well-known logo.



It is one of the curves collectively called Lissajous curves. In this task you are to investigate a few of these curves as shown below. Each curve is a relation in the Cartesian plane.



Assumed knowledge:

Functions and relations; sine and cosine functions; product of functions; transformations; inverse; graphs; algebra; differential and integral calculus; CAS

Part I (75 minutes plus)
Formulating some basic closed Lissajous curves

a. Consider $x = \cos(t)$ where $t \in R$. Given $\sin^2(t) + \cos^2(t) = 1$, find $\sin(t)$ in terms of x.

b. Consider $y = \sin(bt)$ where $b = 1$ and $t \in R$. Express y^2 in terms of x, hence express y in terms of x. State the domain and range of the relation. Sketch a graph of the relation.

c. Consider $y=\sin(bt)$ where $b=2$ and $t\in R$. Given that $\sin(2t)=2\sin(t)\cos(t)$, express y^2 in terms of x, hence express y in terms of x.
Use differentiation to calculate the exact coordinates of the stationary points.
Sketch a graph of the relation. Label the axes intercepts and turning points with coordinates.

d. Consider $y=\sin(bt)$ where $b=3$ and $t\in R$.
Given that $\sin(3t)=\sin(t)\cos(2t)+\cos(t)\sin(2t)$, show that $y^2=\left(4x^2-1\right)^2\left(1-x^2\right)$.
Calculate the exact coordinates of the stationary points.
Sketch a graph of the relation. Label the axes intercepts and turning points with coordinates.

e. Consider $y = \sin(bt)$ where $b = 4$ and $t \in R$.
By writing $\sin(4t)$ as $\sin(2(2t))$ and use some of the results in working out parts a to d, show that $y^2 = 16x^2(2x^2 - 1)^2(1 - x^2)$.

Show that the exact coordinates of the 8 stationary points are $\left(\pm\frac{\sqrt{2\pm\sqrt{2}}}{2}, \pm 1\right)$.

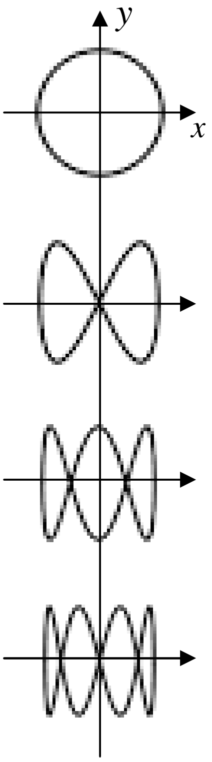
Sketch a graph of the relation. Label the axes intercepts and turning points with coordinates.

f. Relation $y^2 = 16x^2(1 - x^2)(2x^2 - 1)^2$ can be split into two different odd functions, $f(x)$ and $g(x)$, which are continuous and differentiable in the interval (α, β).
State the value of each of α, β and $f(x) + g(x)$.
Select either $f(x)$ or $g(x)$ and show it is an odd function.

g. Select a relation from parts b, c or d, which can be split into two different functions, $h(x)$ and $k(x)$, such that they are continuous and differentiable in the interval (α, β) as determined in part f, and they are reflections (in the x-axis) of each other.
State the equation of your selected relation. State the equation of either $h(x)$ or $k(x)$, and show that it is an even function.

End of Part I

Part II (50 minutes plus)
Further investigation of the basic closed Lissajous curves
If Part I and Part II are done separately, students may use the following information from Part I.



$y = \sin(bt)$ where $b = 1$ and $x = \cos(t)$
Cartesian equation: $y^2 = 1 - x^2$

$y = \sin(bt)$ where $b = 2$ and $x = \cos(t)$
Cartesian equation: $y^2 = 4x^2(1 - x^2)$

$y = \sin(bt)$ where $b = 3$ and $x = \cos(t)$
Cartesian equation: $y^2 = (4x^2 - 1)^2(1 - x^2)$

$y = \sin(bt)$ where $b = 4$ and $x = \cos(t)$
Cartesian equation: $y^2 = 16x^2(2x^2 - 1)^2(1 - x^2)$

a. In terms of integer b, write a formula for the total number N of turning points for the type of Lissajous curves shown above.

b. Discuss the type(s) of symmetry shown by the Lissajous curves above.

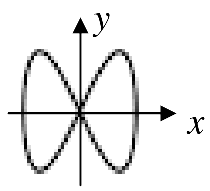
c. For the type of Lissajous curves shown above, each can be split into two different functions as discussed in Part I . Let them be $p(x)$ and $q(x)$. Discuss how the value of integer b can be used to tell whether both are even functions or both are odd functions.

d. Select an integer value of b where $4 < b \le 7$.
Sketch carefully $p(x)$ and $q(x)$ obtained from the relation ($y = \sin(bt)$ and $x = \cos(t)$) for your selected b value on separate set of axes.

e. Determine the area enclosed by each of the above closed Lissajous curves, correct to 4 decimal places.
Briefly discuss the effects of changing the integer b value on the enclosed area.

f. Determine the average value of $p(x)$ (or $q(x)$) over its maximal domain for each of the above closed Lissajous curves, correct to 4 decimal places.
Write a brief statement about the average values of $p(x)$ (or $q(x)$) and the integer b values.

g. For each of the above closed Lissajous curves, write down the equation of its inverse.

h. Express y^2 in terms of x for the equation of the inverse of 

algebraically.

End of Modeling Task