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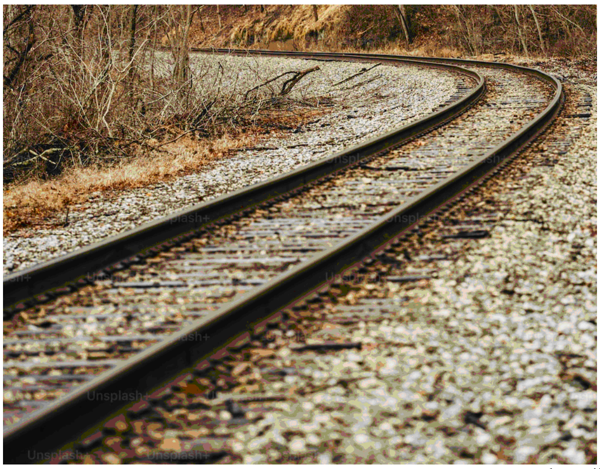
2025 Specialist Mathematics

Year 12 Application Task

(Time allowed: 4 hours plus)

Application Task
Theme: Locus of a point at a constant distance from a smooth curve

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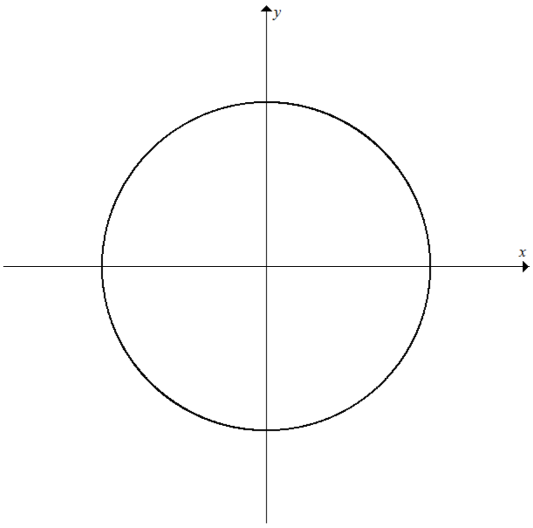
Train tracks are examples of the theme.
Any point on one track (locus) has the same perpendicular distance from the other track (smooth curve).

Assumed knowledge: Coordinate geometry; algebra; functions, relations and graphs; locus of a point; parametric equations; differentiation; length of a curve; CAS

Part I (80 minutes plus) Loci

The following diagram shows a closed curve on a Cartesian plane (a circle of radius 5 units centred at $(0,0)$).



a. On the diagram above sketch two possible loci of a point which is always 1 unit from the circle.

b. Write down the Cartesian equation of each of the two loci.

c. Let A be a point on the given circle, and the x-coordinate of A is p.
B is **a point directly opposite to** A on the outer locus. Show that the x-coordinate and the y-coordinate of point B are $x=\frac{6p}{5}$ and $y=\frac{6\sqrt{25-p^2}}{5}$. They are the parametric equations of the locus of B in terms of parameter p.

d. C is a point directly opposite to A on the inner locus. Find the parametric equations of the locus of C in terms of p.

e. Use the parametric equations of the locus of B to verify it is a circle.

Consider a straight line of equation $y=mx+n$ where $m,n\in R\setminus\{0\}$. Choose a suitable value for each of m and n to specify your line. Point E is below your line and it is 1 unit from your line.

f. Find the Cartesian equation of the locus of E.

g. Let D be a point on your line, and its x-coordinate be λ. Determine the parametric equations of the locus of E with λ as the parameter.

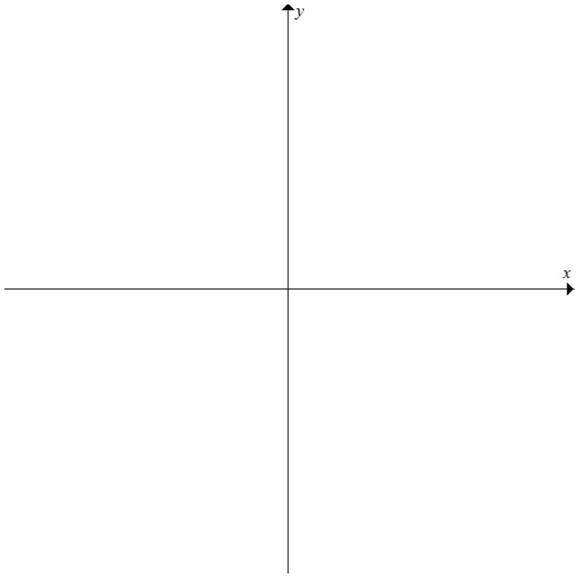
h. Use the parametric equations of the locus of E to verify it is the same line found in part f.

In parts a to e, it appears that the locus of a point at a constant perpendicular distance from a circle is a circle; and in parts f to h, the locus of a point at a constant perpendicular distance from a straight line is a straight line.
Is it valid to make the conclusion that the locus of a point at a constant perpendicular distance from a curve, for example an ellipse, is an ellipse?

In the following parts, you are to investigate the validity of the conclusion for an ellipse of the form $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ where $\alpha,\beta\in R^+$ and $\alpha\neq\beta$.

i. Select a suitable value for each of α and β in $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$.

Using your selected values, sketch ellipses $\varepsilon_1:\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ and $\varepsilon_2:\frac{x^2}{(\alpha+1)^2}+\frac{y^2}{(\beta+1)^2}=1$ on the same Cartesian plane.



j. Determine whether ellipse ε_2 is the locus of a point 1 unit from ellipse ε_1.
Hint: Pick a specific convenient point on ellipse ε_2 away from the axis intercepts. Use calculus methods and by CAS to determine its shortest distance from ellipse ε_1. Comment/discuss your findings.

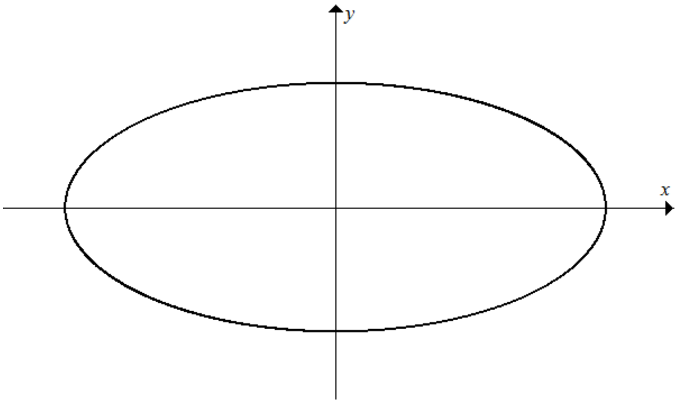
End of Part I

Part II (80 minutes plus) Train tracks

Train tracks are examples of the theme: **Locus of a point at a constant distance from a smooth curve**
Any point on one track (locus) has a constant perpendicular distance from the other track (smooth curve).
Standard gauge train tracks are 1.435 m apart. The linear measurements in **Part II** are in metres (m).

Consider train tracks forming a closed loop. The inner track is an elliptical track given by $\frac{x^2}{20^2}+\frac{y^2}{10^2}=1$.

Any point on the outer track is 1.435 m from the inner track.
The following diagram shows an ellipse representing the inner track of the rail loop.



a. $P\left(p, \sqrt{100-\frac{p^2}{4}}\right)$ is a point on the inner track, where parameter $p \in R$.

State the possible values of p.

Show that the equation of the normal to the ellipse at point P is $y-\sqrt{100-\frac{p^2}{4}}=\frac{4\sqrt{100-\frac{p^2}{4}}}{p}(x-p)$.

The normal at $P\left(p, \sqrt{100-\frac{p^2}{4}}\right)$ intersects the outer track at $Q(x, y)$, and it is perpendicular to the tangent to the outer track at $Q(x, y)$.

b. Write an equation in terms of x, y and p to show $QP = 1.435$ m.

c. Solve the two equations found in parts a and b to show that $x = p\left(1+\frac{1.435}{\sqrt{1600-3p^2}}\right)$ and

$y = \frac{\sqrt{400-p^2}}{2}\left(1+\frac{5.74}{\sqrt{1600-3p^2}}\right)$.

The results found in part c are the parametric equations of the locus of Q which is the outer track of the rail loop.

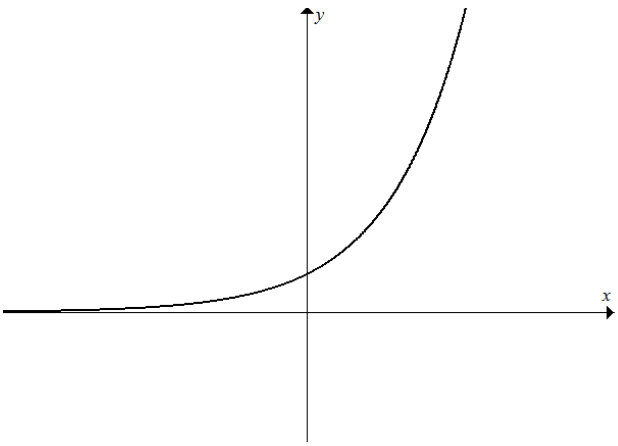
d. Verify that the two tracks are indeed 1.435 m apart by choosing two general points on the inner track, one above and one below the x-axis.

e. Display on your CAS the track (the locus of Q) given by parametric equations $x = p\left(1+\frac{1.435}{\sqrt{1600-3p^2}}\right)$

and $y = \frac{\sqrt{400-p^2}}{2}\left(1+\frac{5.74}{\sqrt{1600-3p^2}}\right)$, together with the ellipse given by $\frac{x^2}{21.435^2}+\frac{y^2}{11.435^2}=1$.

Compare the two graphs and comment. You are not required to sketch the graphs.

Consider a train track given by exponential function $y = e^{kx}$ where $k \in R^+$.
The following diagram shows the graph of a track for a particular k value.



$P\left(p, e^{kp}\right)$ is a point on the track $y = e^{kx}$, where parameter $p \in R$. The other track is below the track shown in the diagram. The two tracks are 1.435 m apart (standard gauge).

f. Choose a suitable value for k and use simular methods employed in parts a, b and c to find the parametric equations of the other track.

g. Verify that the two tracks are indeed 1.435 m apart by choosing two general points on the track $y = e^{kx}$ such that $p < 0$ and $p > 0$, where k is your chosen value.

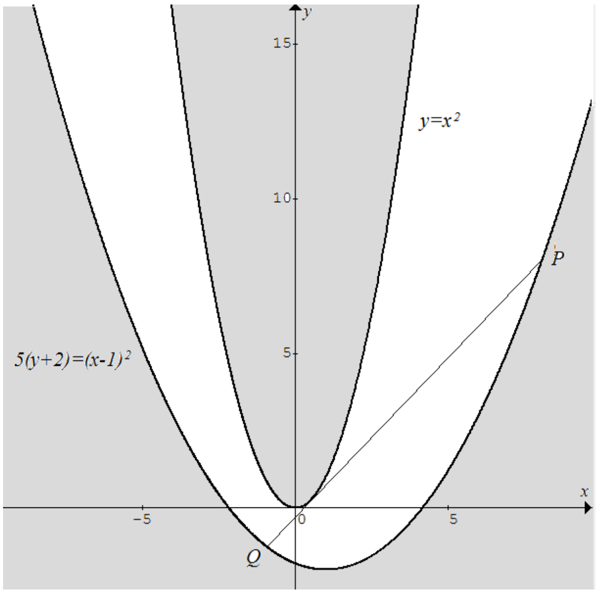
h. Calculate the length (m) of the track $y = e^{kx}$ from $p = -10$ to $p = 5$, correct to 3 decimal places, where k is your chosen value.

i. Calculate the length (m) of the other track from $p = -10$ to $p = 5$, correct to 3 decimal places. Comment on the difference in length of the two tracks.

End of Part II

Part III (80 minutes plus) The longest

A passage is bounded by two parabolas, $y = x^2$ and $5(y+2) = (x-1)^2$ as shown below. Length is in metres. A rigid thin rod represented by line PQ touching both parabolas is also shown.



PQ is a tangent to the parabola at $(p,\ p^2)$.

a. Show that PQ has equation $y = 2px - p^2$.

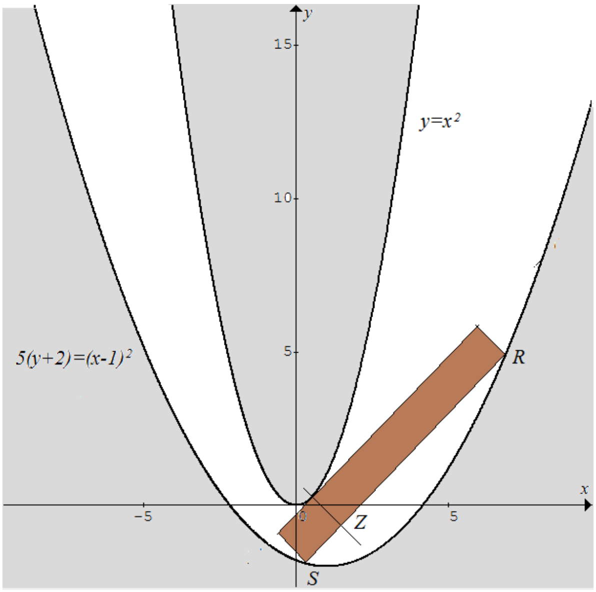
b. Briefly describe the change in distance between P and Q as the value of p decreases.

c. Show that the coordinates of points P and Q in terms of parameter p are:
$P\left(5p+1+\sqrt{20p^2+10p+10},\ 9p^2+2p+2p\sqrt{20p^2+10p+10}\right)$ and
$Q\left(5p+1-\sqrt{20p^2+10p+10},\ 9p^2+2p-2p\sqrt{20p^2+10p+10}\right)$.

d. Let D be the distance from P to Q. Show that $D^2 = 40\left(2p^4 + p^3 + 3p^2 + p + 1\right)$.

e. Find the longest rod PQ that can move around the bend of the passage, correct answer to 4 decimal places.

Now a rectangular plank lies flat in the passage bounded by two parabolas, $y = x^2$ and $5(y+2) = (x-1)^2$ as shown below. The plank has a width of 1 m. Corners *R* and *S* are in contact with parabola $5(y+2) = (x-1)^2$, and the opposite side parallel to *RS* is a tangent to parabola $y = x^2$ at $(p,\ p^2)$.
A normal to parabola $y = x^2$ at $(p,\ p^2)$ intersects *RS* at *Z*.



f. Show that the equation of the normal to parabola $y = x^2$ at $(p,\ p^2)$ is $y = p^2 + \frac{1}{2} - \frac{x}{2p}$.

g. Show that the parametric equations of the locus of point Z are $x = p + \frac{2p}{\sqrt{1+4p^2}}$ and $y = p^2 - \frac{1}{\sqrt{1+4p^2}}$.

h. Show that the equation of edge RS is $y = 2px - p^2 - \sqrt{1+4p^2}$

i. Edge RS intersects parabola $5(y+2) = (x-1)^2$ at R and S. Complete the following table showing the coordinates of R and S, and D the length of RS, for different values of parameter p.

p	$R(x, y)$	$S(x, y)$	D
-0.35			
-0.30			
-0.25			
-0.20			
-0.15			
-0.10			

j. Use the values in the above table to estimate the length (m) of the longest rectangular plank that can move around the bend of the passage. Additional data from q values of your choice may be used.

End of Part III
End of Application Task