2025 Specialist Mathematics

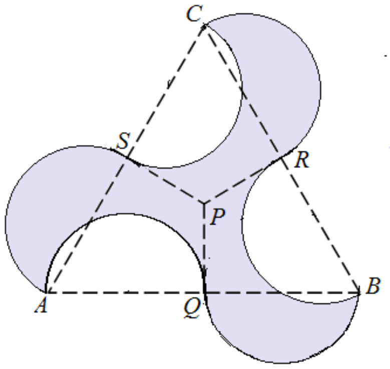
Year 12 Problem Solving Task

(Time allowed: 2.0 hours plus)

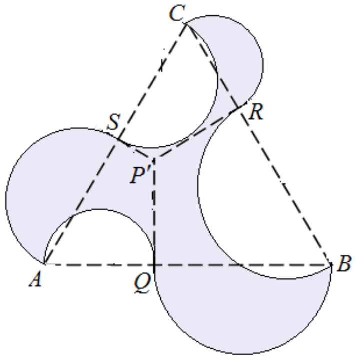
Problem Solving Task

Theme: Area of a irregular 2-dimensional 3-blade ‘propeller’

Information:



Regular



Irregular

In the diagrams above, ABC is an equilateral triangle. The side length is 1 unit.
In the regular propeller, PQ, PR, and PS are *perpendicular bisectors* of sides AB, BC, and CA respectively.
In the irregular propeller, P' is any point inside equilateral triangle ABC.
$P'Q$, $P'R$, and $P'S$ are perpendicular to sides AB, BC, and CA respectively.
In both diagrams of the propellers, the blades are formed with semi-circles.
The semi-circles are the same in the regular propeller, but they are not in the irregular propeller.

The task is to find the area (shaded) of the **irregular** propeller.

Assumed knowledge: Equilateral triangle; geometry; compound angle formulas; area of a semi-circle; area of a triangle; vectors and vector projections; algebra; CAS

Part I (70 minutes plus)

a. Calculate the area of ΔABC.

b. Place ΔABC on a Cartesian plane such that A is at the origin, B is on the positive x-axis, and C is in the first quadrant of the plane. Unit vectors $\tilde{i}$ and $\tilde{j}$ are in the positive x and y directions respectively. Express $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ in terms of $\tilde{i}$ and $\tilde{j}$.

c. Use vector methods to find the area of equilateral ΔABC.

d. Clearly explain why the area (shaded) of the **regular** propeller is equal to the area equilateral ΔABC.

e. Consider any vector $\tilde{p}$ inside ΔABC and $|\tilde{p}| = r$ where $r = \frac{1}{2}$. Let the angle between $\tilde{p}$ and $\overrightarrow{AB}$ be θ. Calculate the scalar resolute (scalar projection) of $\tilde{p}$ in the direction of each of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

f. In terms of r, express the scalar resolute of $\tilde{p}$ in the direction of each of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

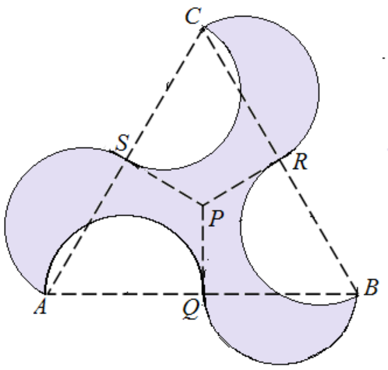
g. Show that the sum of the scalar resolutes in part e is zero, and that for part f is also zero.

h. Write a general statement for part g.

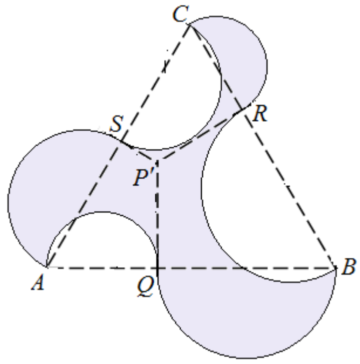
i. Discuss/explain whether your general statement in part h is true for *non-equilateral* triangles.

End of Part I

Information given in Part I:



Regular

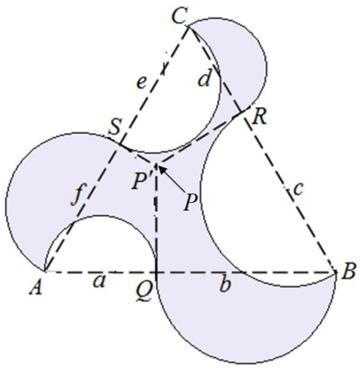


Irregular

In the diagrams above, ABC is an equilateral triangle. The side length is 1 unit.
In the regular propeller, PQ, PR, and PS are *perpendicular bisectors* of sides AB, BC, and CA respectively.
In the irregular propeller, P' is any point inside equilateral triangle ABC.
$P'Q$, $P'R$, and $P'S$ are perpendicular to sides AB, BC, and CA respectively.
In both diagrams of the propellers, the blades are formed with semi-circles.
The semi-circles are the same in the regular propeller, but they are not in the irregular propeller.
The area (shaded) of the **regular** propeller is equal to the area equilateral ΔABC.

Part II (70 minutes plus)

Consider the diagram showing the irregular propeller.
Point P in the following diagram is the same point P as shown in the diagram of the regular propeller,
Vector PP' is added to the diagram.
Let $AQ = a$, $QB = b$, $BR = c$, $RC = d$, $CS = e$ and $SA = f$.
Let $|PP'| = r$ and the angle between $\overrightarrow{PP'}$ and $\overrightarrow{AB}$ be θ.



a. Let the scalar resolute of $\overrightarrow{PP'}$ in the direction of each of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ be α, β and γ respectively. Determine α, β and γ in terms of r and θ.

Show that $r\cos\theta + r\cos\left(\theta - \frac{2\pi}{3}\right) - r\cos\left(\theta - \frac{\pi}{3}\right) = 0$ by using compound-angle formulas.

Note: You obtained the same results by using $\tilde{i}$ and $\tilde{j}$ components in Part I g.

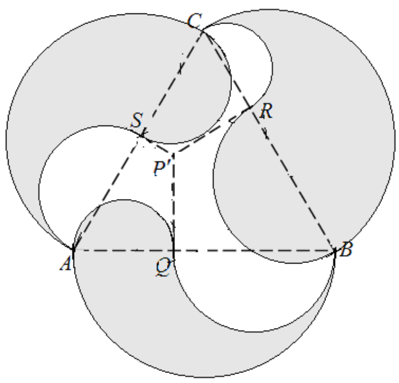
b. Show/explain that $a=\frac{1}{2}+r\cos\theta$, $b=\frac{1}{2}-r\cos\theta$, $c=\frac{1}{2}+r\cos\left(\theta-\frac{2\pi}{3}\right)$, $d=\frac{1}{2}-r\cos\left(\theta-\frac{2\pi}{3}\right)$,
$e=\frac{1}{2}-r\cos\left(\theta-\frac{\pi}{3}\right)$ and $f=\frac{1}{2}+r\cos\left(\theta-\frac{\pi}{3}\right)$

Hence show that $a+c+e=b+d+f=\frac{3}{2}$.

c. Write a statement (in words) expressing the area of the shaded irregular propeller in terms of the *areas of the equilateral* ΔABC and the *semi-circles of diameters* a, b, c, d, e and f.

d. Following your written statement in part c, write a formula expressing the area of the shaded irregular propeller in terms of a, b, c, d, e and f.
Hence find the area of the shaded irregular propeller.

Three more semi-circles were added to form three drops as shown below.

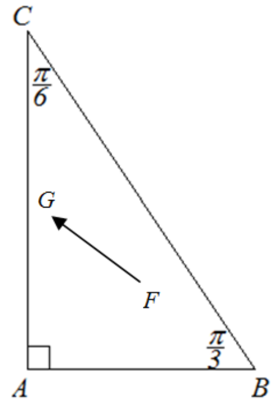


e. Find the total area of the three drops

Now consider irregular propeller from a non-equilateral triangle.

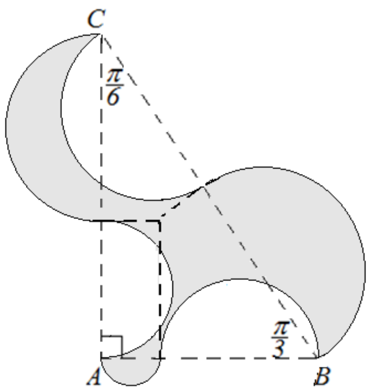
f. In the diagram below, $\overrightarrow{FG}$ is any vector inside right-angle triangle ABC where $\angle ABC = \frac{\pi}{3}$ and $\angle ACB = \frac{\pi}{6}$. $\overrightarrow{FG}$ is not parallel to any one of the three sides. Let $\left|\overrightarrow{FG}\right| = 1$ and the angle between $\overrightarrow{FG}$ and $\overrightarrow{AB}$ be θ.

Select an appropriate value for θ. Determine the sum of the scalar resolutes of $\overrightarrow{FG}$ in the directions of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.



g. Compare the results in part a and part f.

h. The following irregular propeller is formed using the right-angle triangle and six semi-circles.



Explain whether the areas of the irregular propeller (shaded) and the right-angle triangle are equal or not. Fully and concisely summarise your findings in Part I and Part II about the areas of an irregular propeller and the triangle used.

End of Part II

