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2026 Mathematical Methods

Year 12

Application Task

Time allowed: 4 hours plus

Application Task

Theme: Intersections of circles and parabolas

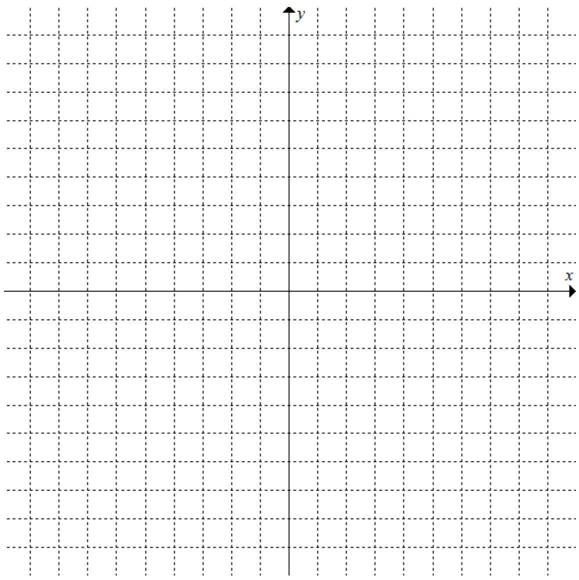
Assumed knowledge:

Functions and relations, domain and range, algebra, quadratic functions and circles, graphs, co-ordinate geometry, parameters, transformations, simultaneous equations, differentiation and integration, CAS

The task: To investigate the conditions for intersection of a circle and a parabola to occur, and the number of intersections possible under certain conditions.

Part I (80 minutes plus)
Circle touching parabola at its vertex

Consider circle $x^2+(y-1)^2=1$ and parabola $y=nx^2$ where $n\in R^+$ is a parameter.
a. Choose a value of $n\le 0.3$ and another value of $n\ge 1.3$.
Sketch the circle and the two parabolas of your chosen n values on the Cartesian plane.
Comment on the intersections of the circle with each parabola, including their coordinates corrected to 4 decimal places.



b. Algebraically, solve $x^2+(y-1)^2=1$ and $y=nx^2$ simultaneously to find the coordinates of the intersections in terms of n.

c. Find the values of n such that there are more than one intersection of $x^2+(y-1)^2=1$ and $y=nx^2$.

d. State the possible values of the y-coordinate of the intersections.

e. Instead of solving simultaneous equations as in part b, use gradient functions to find the values of n such that there are more than one intersection of $x^2+(y-1)^2=1$ and $y=nx^2$.

f. At one of the intersections of the circle and the parabola for $n=1$, find the acute angle between the two curves. Express your answer in simplest exact form.

g. For a particular n value, parabola $y = nx^2$ divides the region inside circle $x^2 + (y-1)^2 = 1$ into three regions of equal area.
Write an equation involving a definite integral, which represents the above statement.
Use CAS to find the value of n, correct to 4 decimal places, Hint: Look for $n > 10$.

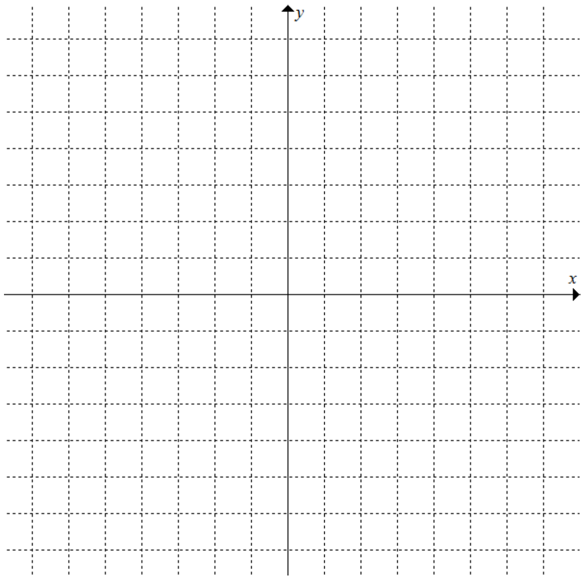
h. At the intersection of circle $x^2 + (y-1)^2 = 1$ and parabola $y = nx^2$, a tangent line to the circle is drawn. Discuss the effect of increasing the n value on the area of the region bounded by the tangent line, the parabola and the x-axis.

i. State the range of the area in part h.

End of Part I

Part II (80 minutes plus)
Circle inside parabola

Consider circle $x^2 + y^2 = r^2$ and parabola $y = x^2 - 1$ where $r \in R^+$ is a parameter.
a. Choose a value of $r \leq 0.8$ and another value of $r \geq 1.5$.
Sketch the two circles and the parabola of your chosen r values on the Cartesian plane.
Find the coordinates of the intersections of the circles with parabola $y = x^2 - 1$, correct to 4 decimal places.



b. State the number of possible intersections of circle $x^2 + y^2 = r^2$ and parabola $y = x^2 - 1$ when parameter r changes.
Draw diagrams to illustrate each possibility. Label each diagram with its possible r value.

c. Algebraically, show that circle $x^2 + y^2 = r^2$ and parabola $y = x^2 - 1$ intersect at (x, y) such that $x^2 = \frac{1}{2}\left(1 - \sqrt{4r^2 - 3}\right)$ or $x^2 = \frac{1}{2}\left(1 + \sqrt{4r^2 - 3}\right)$ for some positive values of r.

d. Determine the range of r values in exact form such that circle $x^2 + y^2 = r^2$ and parabola $y = x^2 - 1$ do not intersect.

e. State the value of r for one intersection if it exists.

f. Determine the r values in exact form such that circle $x^2 + y^2 = r^2$ and parabola $y = x^2 - 1$ have exactly two intersections.

g. Determine the r value such that circle $x^2+y^2=r^2$ and parabola $y=x^2-1$ have exactly three intersections. Find the coordinates of the intersections.

h. Determine the range of r values in exact form such that circle $x^2+y^2=r^2$ and parabola $y=x^2-1$ have exactly four intersections.

i. State the r value such that the upper section of circle $x^2+y^2=r^2$ and the lower section of parabola $y=x^2-1$ will form a smooth closed loop, i.e. the loop is differentiable at the joining points of the circle and the parabola.

j. Determine the area of the region inside the closed loop in part i, correct to 4 decimal places.

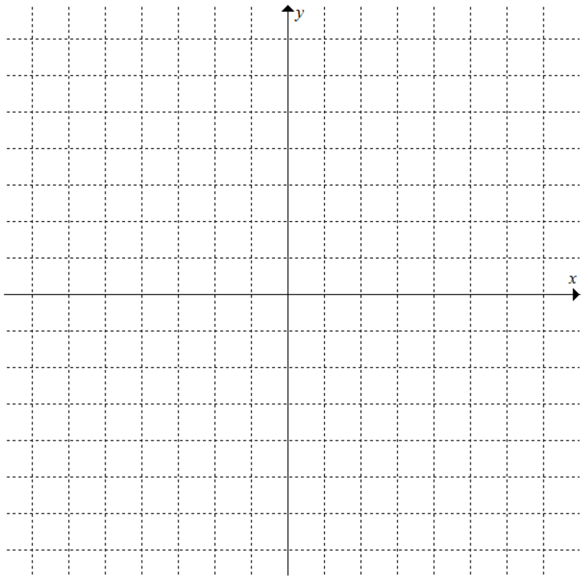
End of Part II

Part III (80 minutes plus)

Separated parabola and circle

Consider circle $x^2 + y^2 = r^2$ and parabola $y = a(x-b)^2 + c$ where $r \in R^+$ and $a, b, c \in R$.

a. Choose a suitable value for each of a, b, c and r such that circle $x^2 + y^2 = r^2$ and parabola $y = a(x-b)^2 + c$ never intersect. Sketch your chosen circle and parabola.



b. For circle $x^2 + y^2 = r^2$ and parabola $y = \frac{1}{8}(x-6)^2$ where $r \in R^+$, investigate the effects on the number of intersections as the r value increases from 0.

Determine the coordinates of the intersections in exact form for one of the intersections.

State the exact r values for the intersections to occur.

c. For circle $x^2 + y^2 = 1$ and parabola $y = a(x-2)^2$ where $a \in R^+$, determine the value of a such that $y = a(x-2)^2$ intersects $x^2 + y^2 = 1$ at exactly one point.
Find the coordinates of the intersection. Correct all answers to 4 decimal places.

d. Choose a suitable value for $a \in R^+$ such that circle $x^2 + y^2 = 1$ is outside parabola $y = a(x-b)^2$. For your chosen value of a, determine the value of b such that $y = a(x-b)^2$ intersects $x^2 + y^2 = 1$ at exactly one point.
Find the coordinates of the intersection. Correct all answers to 4 decimal places.

Consider circle $x^2+(y-1)^2=r^2$ and parabola $y=x^2-2x+4$.

e. Find the shortest distance between the circle and the parabola in terms of r, correct to 4 decimal places.

f. When $r=1$, find the coordinates of the points, one on the circle and one on the parabola, giving the shortest distance. Correct all answers to 4 decimal places.

End of Part III
End of Application Task