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2026 Mathematical Methods

Year 12

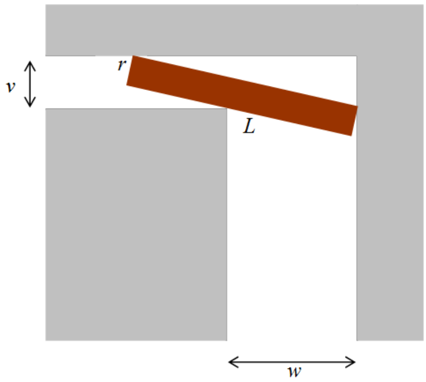
Problem Solving Task

Time allowed: 2 hours plus

Problem Solving Task

Theme: The longest and the largest rectangles around a corner

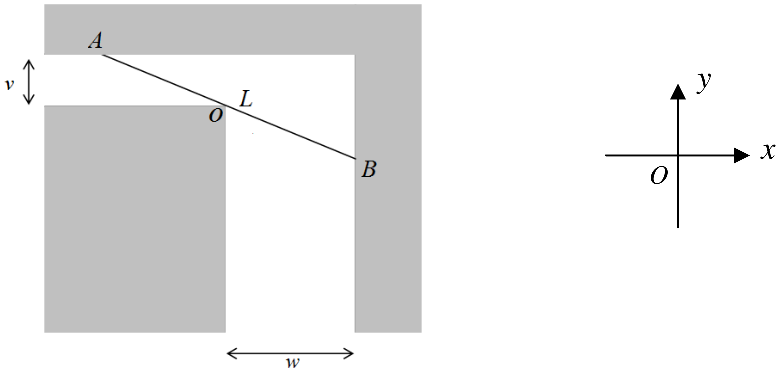
Introduction: Given two straight passages meeting at right angle, the ability of moving a rectangular object from one passage to another passage depends on its dimensions (length L and width r) and the width of each passage, v, w.



Assumed knowledge: Coordinate geometry; algebra; functions, relations and graphs; calculus; CAS

Part I (75 minutes plus)

Consider an object with zero width, a line segment, just fits in the corner.
O is the origin of the Cartesian plane.



a. Let $v = 1$ and $w = 2$.
Write down the coordinates of point A and point B in terms of the gradient m of line AB.

b. Show that distance $AB = \sqrt{4m^2 - 4m + 5 - \frac{4}{m} + \frac{1}{m^2}}$.
Determine the longest line segment L which can move around the corner of the passages.

c. Let $v=1$ and $w=t$. Write down the coordinates of point A and point B in terms of t and the gradient m of line AB.

d. Express distance AB in terms of t and m.

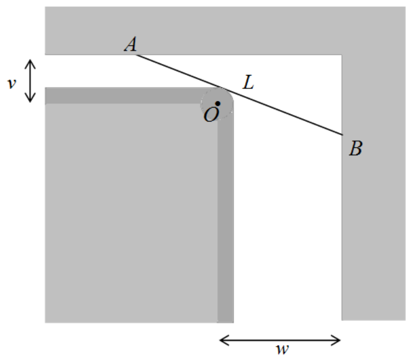
Show that the line segment L is longest when $t^2m^4+tm^3+tm-1=0$. Show that $m=-\frac{1}{\sqrt[3]{t}}$.

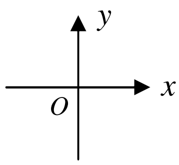
e. Sketch the graph of m versus t for the longest line segment L.
Describe the variation of m with t.

f. Write down the length of the object which can move through the corner of the passages. Explain.

The sharp corner is changed to a circular section centred at origin O with a radius of $\frac{1}{2}$.

The passages become narrower.





g. Write down the equation of the circular section in the first quadrant of the Cartesian plane, including its domain.

h. Find the equation of the tangent to the circular section at $x = a$, $0 \le a \le \frac{1}{2}$.

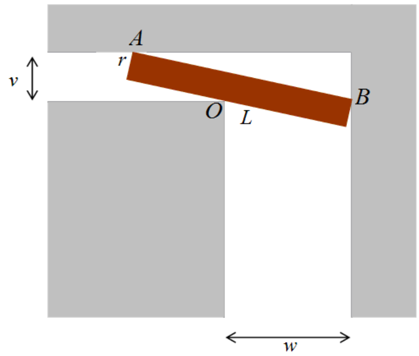
Let $v = 1$ and $w = 2$.
Find the coordinates of point A and point B and the distance AB in terms of a.

i. Determine the longest line segment L which can move through the rounded corner of the passages.

End of Part I

Part II (65 minutes plus)

Given two straight passages meeting at right angle, the ability of moving a rectangular object from one passage to another passage depends on its dimensions (length L and width r) and the width of each passage, v, w. *It appears as shown below at some stage of its motion.*



The task is to find the largest (in area) rectangular object which can move around the corner.

a. Let $r=\frac{1}{2}$, $v=1$ and $w=2$. State the perpendicular distance of line segment AB from O when the rectangular object moves around the sharp corner.

b. Let P be a point on AB such that $OP \perp AB$ when the rectangular object moves around the corner. Explain and show that the equation of the path of point P is $y=\frac{\sqrt{1-4x^2}}{2}$.
State the range of value for each of x and y in the equation.

c. The task of finding the largest object which can move through the rounded corner is the same as the task of finding the longest line segment in Part I i.
Explain why the two tasks are the same. What is the area of the largest rectangular object with $r=\frac{1}{2}$, $v=1$ and $w=2$.

Now consider a rectangular object with width r.

d. Show that the equation of the path of P is given by $y=\sqrt{r^2-x^2}$, and the tangent to the path at $\left(a, \sqrt{r^2-a^2}\right)$ is $y=\dfrac{r^2-ax}{\sqrt{r^2-a^2}}$.

e. Let $v=1$ and $w=2$. Show that $A\left(\dfrac{r^2-\sqrt{r^2-a^2}}{a},\ 1\right)$ and $B\left(2,\ \dfrac{r^2-2a}{\sqrt{r^2-a^2}}\right)$.

f. Show that distance $AB=\dfrac{r\left(2a-r^2+\sqrt{r^2-a^2}\right)}{a\sqrt{r^2-a^2}}$.

Area of the rectangular object is $\mathcal{A}=\dfrac{r^2\left(2a-r^2+\sqrt{r^2-a^2}\right)}{a\sqrt{r^2-a^2}}$.

Take note that the area $\mathcal{A}$ of the rectangular object varies with r and a.

g. Select an appropriate r value and sketch a graph of $\mathcal{A}$ versus a. Find the largest rectangular object (in area) which can move around the sharp corner O as shown in the opening diagram for your selected r value.

h. Sketch graphs for other r values systematically, observe the trend and determine the area of the largest rectangular object which can move around the sharp corner O as shown in the opening diagram.
State the values of a and r when the rectangular object is the largest in area.

End of Problem Solving Task

