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2027 Mathematical Methods

Year 12

Application Task

Time allowed: 4 hours plus

Application Task

Theme: Coast to coast

Use quadratics, exponential and logarithmic functions to model routes and coastlines.

Assumed knowledge:

Functions and relations, algebra, quadratic, exponential and logarithmic functions, circles, graphs, co-ordinate geometry, parameters, transformations, differentiation and integration, tangent and normal to a curve, and CAS

Part I (80 minutes plus) Correct all numerical values to 4 decimal places unless exact values are required.

A coastline is drawn on the grid with *x*-*y* axes as shown below.
Equation of the coastline is $y = e^{kx}$ where $k \in R$.
Distance is measured in arbitrary unit, and *y*-axis is pointing to the north N.
A tangent to the coastline at $(0, 1)$ with gradient of 1 is pointing to the NE (i.e. N 45º E) direction.

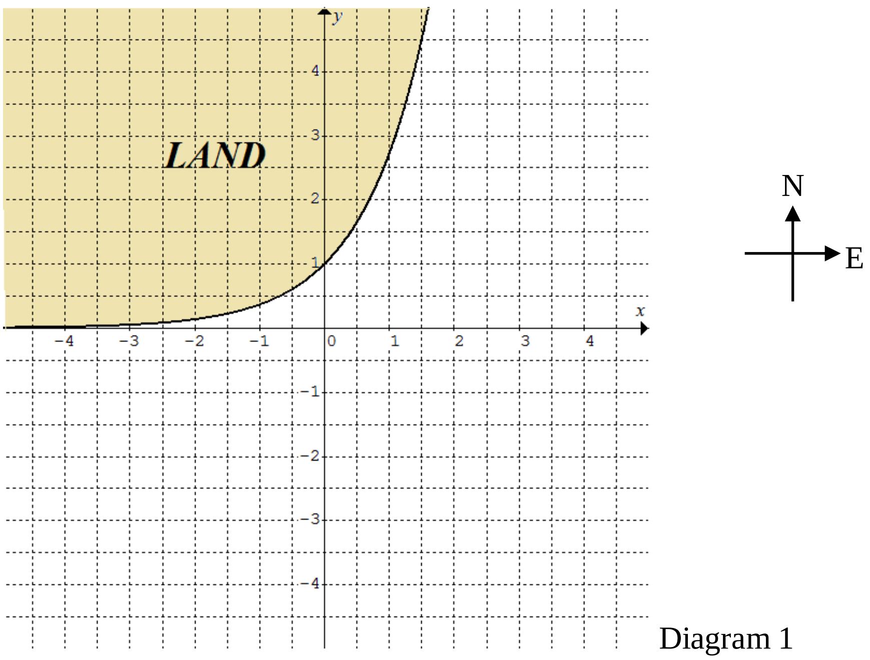

Diagram 1

a. Determine the value of k. State the domain and range of $y = e^{kx}$.

b. Select a point on the coastline. **Calculate** the gradient of the tangent at your selected point and the direction (correct to the nearest degree) of the tangent. Hint: $\tan\theta° = \text{gradient}$

c. Suppose there is another coastline which can be modelled by an exponential function similar to the one in part a.
Write down the equation of this coastline which has an asymptote $y = 1$ and cuts the y-axis at $(0, 2)$ such that the direction of the tangent at the y-intercept is NE.

d. A third coastline can also be modelled by an exponential function.
Write down the equation of this coastline which has an asymptote $y = 0$ and cuts the y-axis at $(0, 2)$ such that the direction of the tangent at the y-intercept is NE.

The opposite shore to the land shown in the given diagram above (Diagram 1) has a coastline with equation of the inverse function of e^{kx}.

e. Write down the equation of the opposite shore coastline. State the asymptote and axis intercept.

f. Select a few representative points on the opposite shore and find their coordinates. Plot these points on the diagram below (Diagram 2) and draw accurately a smooth curve to represent the opposite coastline.

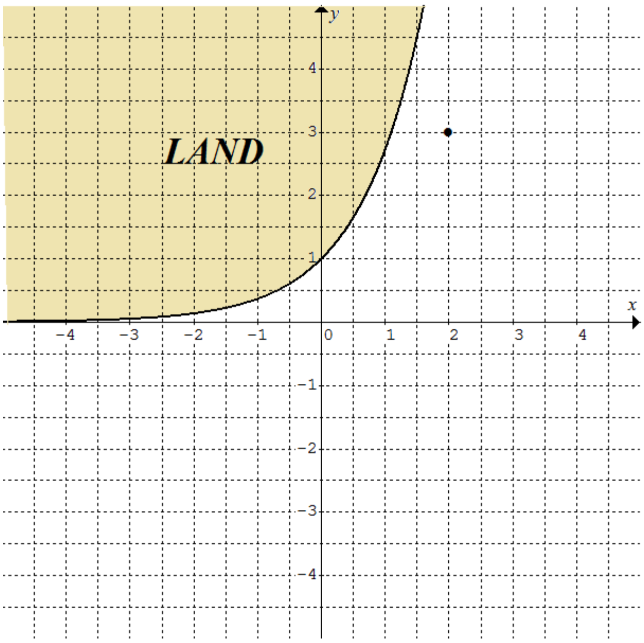

Diagram 2

g. Determine the shortest distance between the two coastlines.

There is an island (considered as a point) with coordinates $(2, 3)$ between the two coastlines.

h. Calculate the shortest distance of the island from each coastline.

End of Part I

Part II (80 minutes plus) Correct all numerical values to 4 decimal places unless exact values are required.

Both coastlines are shown in the accurate diagram below (Diagram 3). The dot at $(2, 3)$ represents the island with insignificant land area.

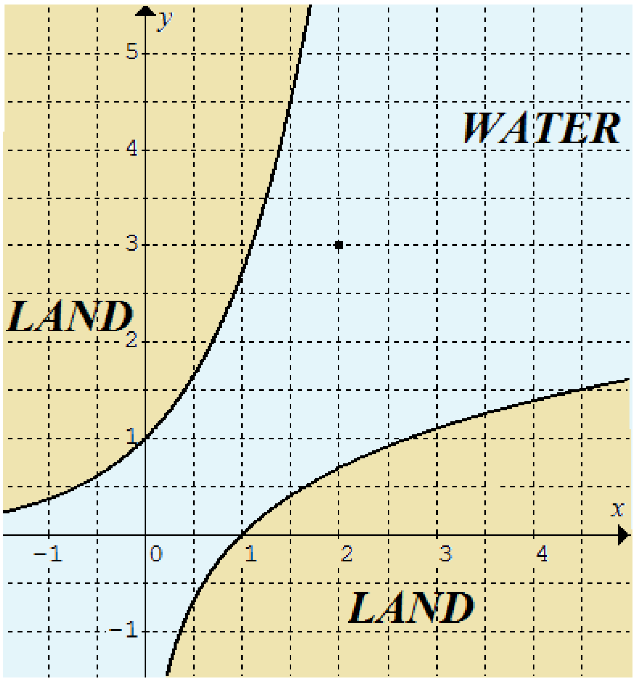

Diagram 3

A horizontal NW straight bridge is to be built connecting the two coastlines over the island at $(2, 3)$.

a. Show that the equation of the bridge is $y = 5 - x$. Determine the length of the bridge from coastline to coastline by firstly finding the coordinates of the start/end point on each coastline.

b. Draw several straight lines through the point $(2, 3)$ from coastline to coastline on Diagram 3.
Graphically decide on the shortest one by measuring with a ruler.
Find the length of the 'shortest' bridge possible based on your measurements.

A cruise ship company runs daily cruise departing from coast town at $(0, 1)$.
The charted cruise route is shown below.
The forward and return routes are parabolic shapes with both vertices at $(0, 1)$.
The route around the island is circular centred at $(2, 3)$. See Diagram 4 below.

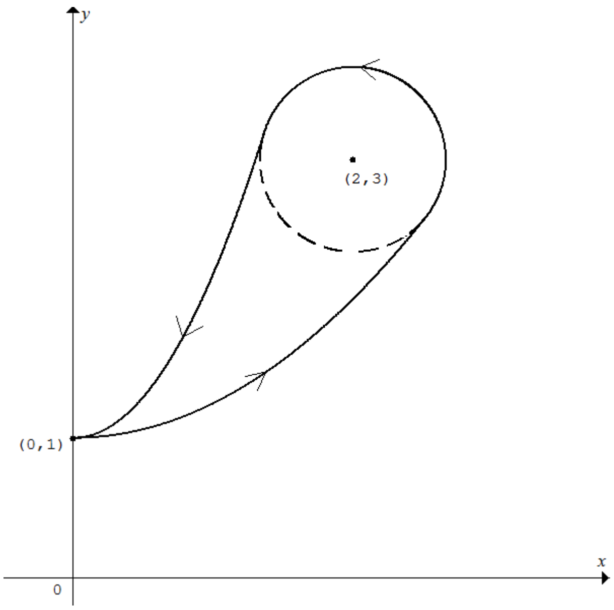

Diagram 4

c. In terms of radius r of the circular route, write down the equation of the circular route.

d. State the transformations to $y = x^2$, in terms of η if necessary, required to give the equation of the forward route $y = \eta x^2 + 1$.

The company brochure shows that at **exactly** one point (p, q) of the forward route the cruise ship is equidistant (same distance) from both coastlines.

e. Determine the value of each of p and q.

f. Determine the distance of (p, q) from either coastline.

g. Show that $\eta = \frac{1}{4}$ in the equation $y = \eta x^2 + 1$ of the forward route.

The company brochure also shows that the whole route after departure from $(0, 1)$ is smooth (i.e. no sudden or abrupt change in direction at the joining points of the different route sections).

h. The forward route and the circular route are joined at point (a, b).
Determine the coordinates of the point (a, b), and show the radius of the circular route is $r \approx 0.66368$.

i. The circular route and the return route are joined at point (c, d).
Determine the coordinates of the point (c, d), and the equation of the return route.

End of Part II

Part III (80 minutes plus) Correct all numerical values to 4 decimal places unless exact values are required.

In Diagram 5 below, the line through point $P(\alpha, \beta)$ and point $(0, 1)$, and the line through point $Q(\mu, \nu)$ and point $(1, e)$ are perpendicular to the coastline $y = e^x$.

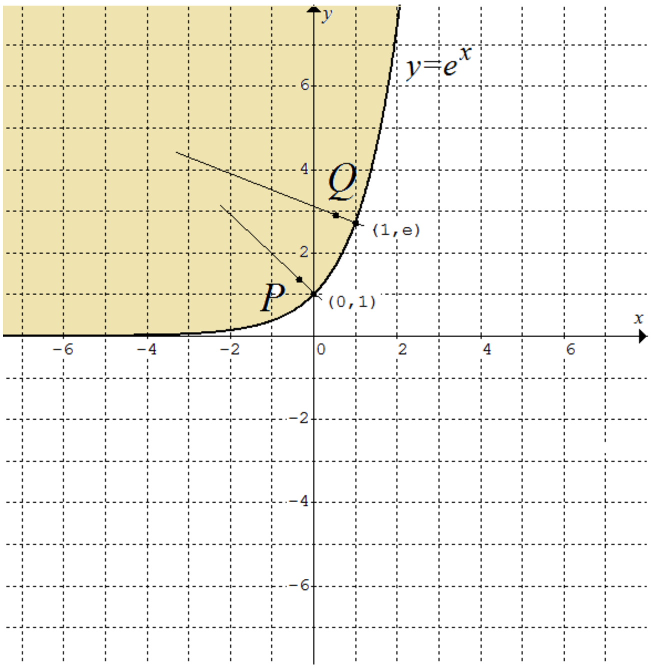

Diagram 5

a. Show that the line through $P(\alpha, \beta)$ is $y = 1 - x$, and the line through $Q(\mu, \nu)$ is $y = e + \frac{1}{e}(1 - x)$.

Both $P(\alpha, \beta)$ and $Q(\mu, \nu)$ have the same perpendicular distance of $\frac{1}{2}$ from the coastline.

b. Show that $(\alpha, \beta) = \left(-\frac{1}{2\sqrt{2}}, 1 + \frac{1}{2\sqrt{2}}\right)$, and $(\mu, \nu) \approx (0.530746, 2.890911)$.

c. A curve of the form $y = \dfrac{\varepsilon}{x - \delta}$ is used as a boundary separating two regions, where ε and δ are constants to be determined. The land between the curve and the coastline is designated as the coastal region. The curve passes through $P(\alpha, \beta)$ and $Q(\mu, \nu)$. Show that $\varepsilon \approx -2.25077$ and $\delta \approx 1.30931$.

d. Discuss quantitatively whether the effects on the boundary curve $y = \dfrac{\varepsilon}{x - \delta}$ are significant when $\varepsilon = -2.25$ and $\delta = 1.31$ are used.

e. On the opposite shore across the water, there is also a coastal region defined by a curve which is the inverse of $y = \dfrac{\varepsilon}{x - \delta}$. Find the equation of the inverse.

f. The coastal regions in part c and part d are flat and level.

(i) Find the area of the coastal region in part c bounded by $x = -5$ and $y = 6$, shaded darker in Diagram 6 below.

(ii) Find the area of the coastal region in part e bounded by $y = -5$ and $x = 6$.

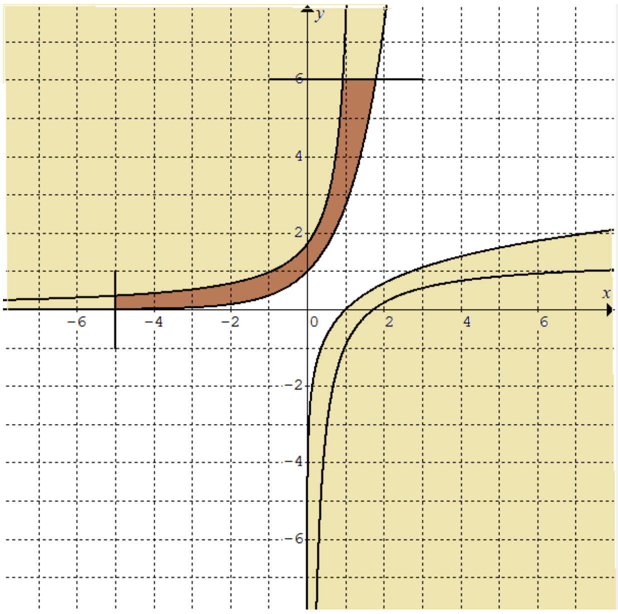

Diagram 6

g. Determine the surface area of the water between the two coastlines bounded by $x = -5$, $y = -5$, $x = 6$ and $y = 6$.

End of Part III