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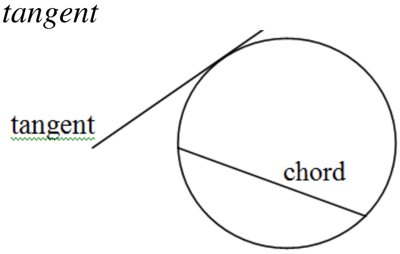
Math Lesson (Suitable for Years 8 to 10)

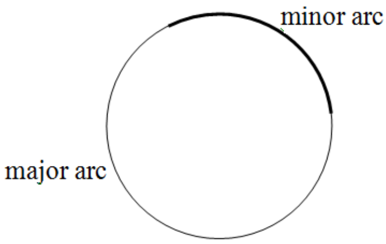
Geometric properties of chords, tangents and angles in circles

Terminology

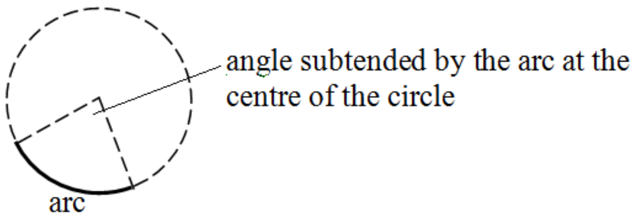
Chord and tangent

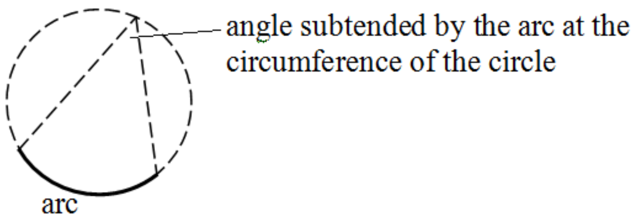


Major and minor arcs



Angles subtended by an arc at the centre and at the circumference

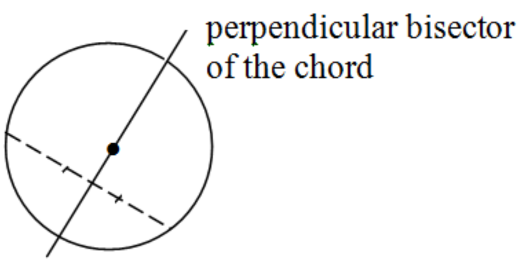




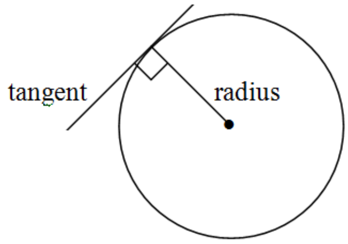
Geometric properties

Some of the following properties are stated without proofs. They are left for students to do in the exercise.

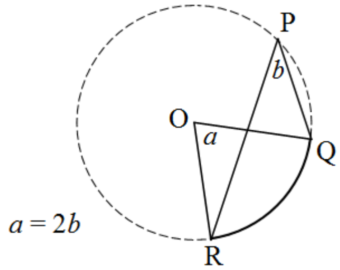
(1) *The perpendicular bisector of a chord passes through the centre of the circle*



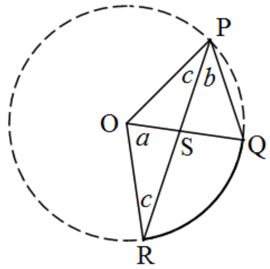
(2) *A tangent is perpendicular to the radius of the circle*



(3) *The angle subtended by an arc at the centre is twice the size of the angle on the circumference*



Proof: Draw line OP.



OP, OQ and OR are radii and have the same length

$\therefore \Delta$OPQ and ΔOPR are isosceles

Let $\angle$OPR = c

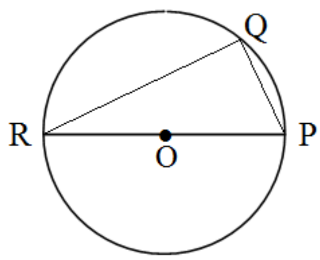
$\therefore \angle$ORP = c and $\angle$OQP = $\angle$OPQ = $b + c$

Now consider ΔOSR and ΔPSR

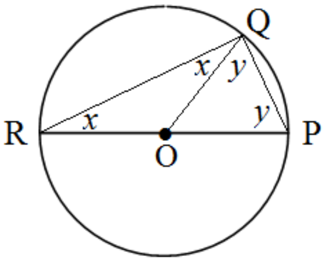
$\angle$OSP is the exterior angle of both triangles

$\therefore$ $a + c = b + (b + c)$, hence $a = 2b$.

(4) *The angle subtended at the circumference by the diameter is a right angle*



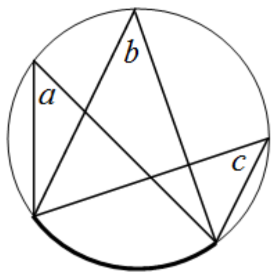
Proof: Draw radius OQ.



$OP = OQ = OR$, $\therefore \Delta OPQ$ and ΔOQR are isosceles.
Let $\angle OQR = \angle ORQ = x$ and $\angle OPQ = \angle OQP = y$
$\therefore \angle POQ = 2x$ (exterior angle)
and $\angle QOR = 2y$ (exterior angle)
$\therefore\ 2x + 2y = 180^o$ (straight angle),
$\therefore\ x + y = 90^o$, $\therefore \angle PQR$ is a right angle.

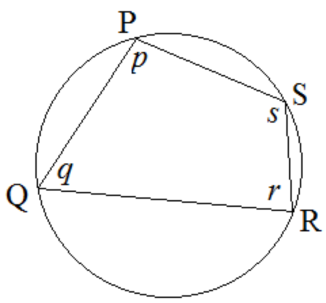
(5) *All angles subtended by the same arc on the circumference are equal*

$a = b = c$

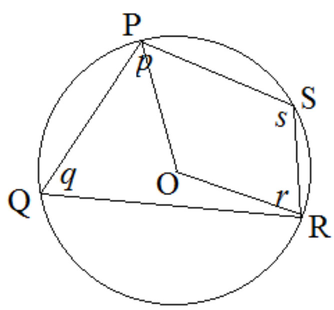


(6) *The opposite angles of a cyclic quadrilateral add up to 180^0, i.e. they are supplementary angles*

A cyclic quadrilateral has all vertices on the circumference. $p + r = 180^o$, $q + s = 180^o$



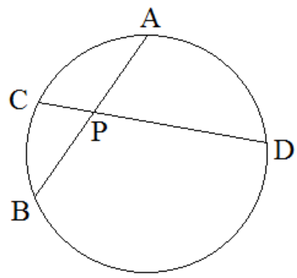
Proof: Draw radii OP and OR.



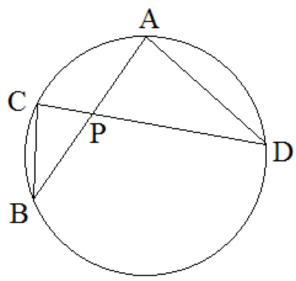
$\angle POR = 2q$ (Property no. 3)
Reflex $\angle POR = 2s$ (Property no. 3)
Since $\angle POR$ + reflex $\angle POR = 360^o$ (1 revolution)
$\therefore 2q + 2s = 360^o$ $\therefore q + s = 180^o$

Similarly, opposite angles $\angle QPS$ and $\angle QRS$ can be shown to be supplementary angles, i.e. $p + r = 180^o$

(7) *For any two intersecting chords, the product of the line segments of one chord equals the product of the line segments of the second chord*
i.e. $\overline{AP} \times \overline{PB} = \overline{CP} \times \overline{PD}$



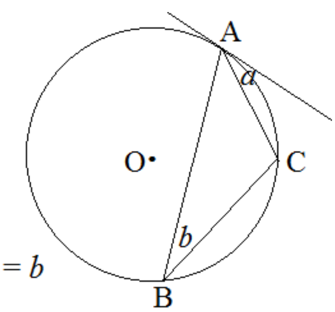
Proof: Draw lines BC and AD.



$\angle BCD = \angle BAD$ because both are subtended by the same arc BD.
$\angle CBA = \angle CDA$ because both are subtended by the same arc AC.
$\angle BPC = \angle APD$ because they are vertically opposite angles. $\therefore \Delta PBC$ and ΔPDA are similar, hence

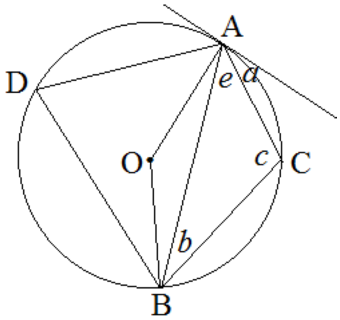
$$\frac{\overline{AP}}{\overline{CP}} = \frac{\overline{PD}}{\overline{PB}} \quad \text{or} \quad \overline{AP} \times \overline{PB} = \overline{CP} \times \overline{PD}$$

(8) *The angle between a tangent and a chord equals the angle subtended by the chord at the circumference in the other segment of the circle.*



$a = b$

Proof: Draw radii OA and OB, chords DA and DB.
Let $\angle BAC = e$ and $\angle ACB = c$.



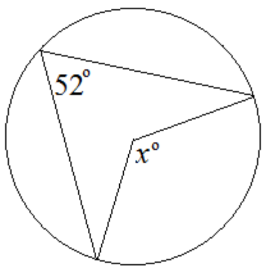
$\angle ADB + c = 180^o$ (opposite angles in a cyclic quadrilateral)
$b + c + e = 180^o$, $\therefore \angle ADB = b + e$
$\angle AOB = 2\angle ADB$ (Property no. 3) $= 2(b + e)$.

$\therefore \angle OAB = \frac{1}{2}(180^o - 2(b + e)) = 90^o - (b + e)$

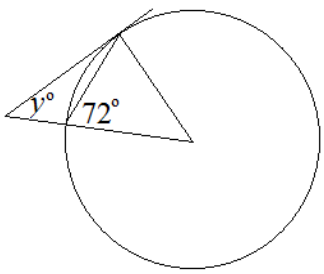
The angle between OA and the tangent $= 90^o$ (Property no. 2) $\therefore \angle OAB + e + a = 90^o$
i.e. $90^o - (b + e) + e + a = 90^o$, $\therefore a = b$
Therefore the angle between a tangent and a chord equals any angle subtended by the chord at the circumference.

Exercise

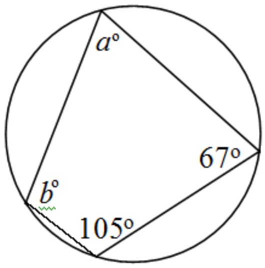
(1) Find the value of x.



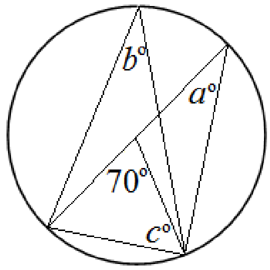
(2) Find the value of y.



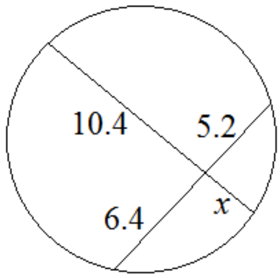
(3) Find the values of a and b.



(4) Find the values of a, b and c.



(5) Find the value of x.



(6) Prove Geometric property 1 on page 1.
Hint: Let the line through the centre of the circle bisect the chord, prove that it is perpendicular to the chord.

(7) Prove Geometric property 5 on page 2.
Hint: Make use of the result in Geometric property 3.

(8) Prove that $a = b$.