Math Lesson (Suitable for Years 11 and 12)
Calculation of uncertainties

In scientific investigations uncertainties are integral parts of all measured and calculated quantities.

An uncertainty in a measured or calculated quantity is expressed in one significant figure and its place value corresponds to that of the last significant figure of the quantity.

In general, for a quantity consisting of the sum and/or difference of many measurements, the uncertainty in the quantity is always the sum of the individual uncertainties.

Example 1 A student used a metre-ruler to measure the height of a building estimated to be between 4 and 5 metres tall. The height was measured again with a measuring tape. The measurements are shown below.

Using a metre-ruler:

$\longleftrightarrow$ 1.00 cm $\longleftrightarrow$ 1.00 cm $\longleftrightarrow$ 1.00 cm $\longleftrightarrow$ 1.00 cm $\longleftrightarrow$ 0.67 cm

With the metre-ruler five measurements were made and the uncertainties of the readings were estimated to be 0.02 m, 0.03 m, 0.03 m, 0.03 m and 0.02 m.

Uncertainty in the measurement of the height is given by the sum of the individual uncertainties, 0.13 m, and rounded ***up*** to one significant figure, i.e. 0.2 m.

.: the height of the building
$\approx 1.00 + 1.00 + 1.00 + 1.00 + 0.67 = 4.67$ m, and including the uncertainty, the height $\approx 4.7 \pm 0.2$ m.

The height was rounded to two significant figures so that the place value of the last figure coincided with that of the uncertainty.

Using a measuring tape: 4.68 m

With a measuring tape only one reading was made and the uncertainty was estimated to be 0.02 m.
.: the height of the building $\approx 4.68 \pm 0.02$ m.

A single measurement is more accurate than the sum of several measurements.

Uncertainties can also be expressed as percentages.

$$\% \text{ uncertainty} = \frac{\text{uncertainty}}{\text{measurement}} \times 100\%$$

Example 2 Express the uncertainty of the height of the building as a percentage.

With the measuring tape:

$$\frac{0.03}{4.68} \times 100\% \approx 0.7\% .$$

With the metre-ruler:

$$\frac{0.2}{4.7} \times 100\% \approx 5\% .$$

For a calculated quantity consisting of the product and/or quotient of two or more quantities, the percent uncertainty is the sum of the percent uncertainties of the individual quantities.

Example 3 A journey consists of 3 *trips* of the same distance 123.9 ± 0.2 km.
The time taken to complete the journey is 5.3 ± 0.1 hours.
Calculate the average speed in kmh^{-1} (including uncertainty).

Uncertainty in distance $= 3 \times 0.2 = 0.6$ km

% uncertainty in distance $= \frac{0.6}{123.9} \times 100\% = 0.5\%$

Uncertainty in time $= 0.1$ h

% uncertainty in time $= \frac{0.1}{5.3} \times 100\% = 2\%$

Therefore % uncertainty in average speed
$= 0.5\% + 2\% = 2.5\%$

Average speed $= \frac{3 \times 123.9}{5.3} = 7.0 \times 10^1$ kmh^{-1}

Uncertainty in average speed
$= 7.0 \times 10^1 \times 2.5\% = 2$ (rounded up to one significant figure)

Average speed $= 70 \pm 2$ kmh^{-1}

Uncertainty in function of a quantity

Uncertainty in the value of a function can be determined using linear approximation,
e.g. if $y = f(x)$, then $\Delta y \approx \Delta x \times f'(a)$ when $x = a$, where Δy is the uncertainty in $f(x)$ for an uncertainty Δx in the quantity x.

Example 4 Determine the volume of a sphere with radius 20.3 ± 0.1 cm (including uncertainty).

$V(r) = \frac{4}{3}\pi r^3$, $V'(r) = 4\pi r^2$

$\Delta V \approx \Delta r \times V'(a) = (0.1)4\pi(20.3)^2 = 6\times10^2 \text{ cm}^3$

$V = \frac{4}{3}\pi(20.3)^3 = 3.50\times10^4 \text{ cm}^3$

Therefore $V = (3.50 \pm 0.06)\times10^4 \text{ cm}^3$

Example 5 Determine the side length of a square with area $7.5 \pm 0.3\,\text{cm}^2$ (including uncertainty).

$l(A) = \sqrt{A}$, $l'(A) = \frac{1}{2\sqrt{A}}$

$\Delta l \approx \Delta A \times l'(a) = 0.3 \times \frac{1}{2\sqrt{7.5}} = 0.06 \text{ cm}$

$l = \sqrt{7.5} = 2.7$ cm (2 significant figures, same as A) and therefore Δl is rounded up to 0.1 cm

$\therefore l = 2.7 \pm 0.1$ cm

Example 6 Given $y = 2\log_e x$, determine the value of y when $x = 12.5 \pm 0.2$ (including uncertainty).

$y = f(x) = 2\log_e x$, $\frac{dy}{dx} = f'(x) = \frac{2}{x}$

$\Delta y \approx \Delta x \times f'(a) = 0.2 \times \frac{2}{12.5} = 0.04$

$y = 2\log_e 12.5 = 5.05$ (same number of significant figures as 12.5)

$\therefore y = 5.05 \pm 0.04$

Example 7 Given $y = 5\sin x$, determine the value of y when $x = 2.5 \pm 0.2$ (including uncertainty).

$y = f(x) = 5\sin x$, $\frac{dy}{dx} = f'(x) = 5\cos x$

$\Delta y \approx \Delta x \times f'(a) = 0.2 \times 5\cos(2.5) = -0.9$ (round up to 0.9 instead of 0.8)

$y = 5\sin(2.5) = 3.0$

$\therefore y = 3.0 \pm 0.9$

Exercise (Include uncertainty in each case)

(1) An object moves in a straight line with the following displacement: 5.25 ± 0.02 m left, 0.78 ± 0.01 m right and 2.37 ± 0.01 m right. Find the total displacement.

(2) A cuboid has the following measurements: $l = 70.8 \pm 0.2$ cm, $w = 15.5 \pm 0.1$ cm, $h = 9.8 \pm 0.1$ cm Calculate its volume in cm^3.

(3) Refer to (1)
Determine the average velocity if the total time taken for the three displacements was 5.3 ± 0.3 s, given average velocity $= \frac{\text{total displacement}}{\text{total time taken}}$.

(4) Given $x = 10e^t$, find x when $t = 0.86 \pm 0.02$

(5) Given $h = 3\cos t$, find h when $t = 3.5 \pm 0.1$