Math Lesson (Suitable for Years 8 to 10)
Surds

Irrational numbers involving radical (root) signs $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$ etc, are called **surds**.

Example 1 $\sqrt{12}$, $\sqrt[3]{4}$ are surds but $\sqrt{9}$, $\sqrt[3]{8}$ are *not* because they are *not* irrational numbers.

$\sqrt{9}=3$ and $\sqrt[3]{8}=2$.

Multiplication and division of surds

Example 2 Simplify (a) $\sqrt{3}\times\sqrt{7}$ (b) $2\sqrt{5}\times3\sqrt{10}$ (c) $\sqrt{6}\div\sqrt{2}$ (d) $\dfrac{3\sqrt{15}}{15\sqrt{3}}$

(a) $\sqrt{3}\times\sqrt{7}=\sqrt{3\times7}=\sqrt{21}$

(b) $2\sqrt{5}\times3\sqrt{10}=2\times3\times\sqrt{5\times10}=6\sqrt{50}$

(c) $\sqrt{6}\div\sqrt{2}=\sqrt{6\div2}=\sqrt{3}$

(d) $\dfrac{3\sqrt{15}}{15\sqrt{3}}=\dfrac{3}{15}\sqrt{\dfrac{15}{3}}=\dfrac{1}{5}\sqrt{5}$ or $\dfrac{\sqrt{5}}{5}$ or $\dfrac{1}{\sqrt{5}}$

Entire surds and standard form

$\sqrt{243}$ and $9\sqrt{3}$ are equal surds. The former is called an entire surd and the latter is written in standard form.

$3\sqrt{27}$ is also equal to $\sqrt{243}$ and $9\sqrt{3}$, but it is neither an entire surd nor a surd in standard form.

A surd in standard form has the number inside the square (or cube etc) root sign not divisible by a perfect square (or cube etc) number greater than 1.

Example 3 Express in standard form.
(a) $\sqrt{125}$ (b) $5\sqrt{32}$

(a) $\sqrt{125}=\sqrt{25\times5}=\sqrt{25}\sqrt{5}=5\sqrt{5}$

(b) $5\sqrt{32}=5\sqrt{16\times2}=5\sqrt{16}\sqrt{2}=20\sqrt{2}$

Example 4 Express as entire surd.
(a) $7\sqrt{5}$ (b) $^{-}5\sqrt{8}$ (c) $9\sqrt{2x}$

(a) $7\sqrt{5}=\sqrt{49}\sqrt{5}=\sqrt{49\times5}=\sqrt{245}$

(b) $^{-}5\sqrt{8}={}^{-}\sqrt{25}\sqrt{8}={}^{-}\sqrt{25\times8}={}^{-}\sqrt{200}$

(c) $9\sqrt{2x}=\sqrt{81}\sqrt{2x}=\sqrt{81\times2x}=\sqrt{162x}$

The multiplication or division of surds can sometimes result in a rational number, e.g.

$3\sqrt{5}\times2\sqrt{125}=6\sqrt{625}=150$, $\dfrac{2\sqrt{18}}{3\sqrt{8}}=\dfrac{2\times3\sqrt{2}}{3\times2\sqrt{2}}=1$

Addition and subtraction of like surds

Like surds have the same radical factor when they are expressed in standard form, e.g. $\sqrt{7}$, $5\sqrt{7}$ and $^{-}2\sqrt{7}$ are like surds because they all have the same radical factor $\sqrt{7}$.

Change surds to standard form in order to determine whether they are like or unlike surds.

Only like surds can be added or subtracted.

Example 5 Are $\sqrt{12}$ and $2\sqrt{75}$ like surds?

Expressing in standard form, $\sqrt{12}=4\sqrt{3}$ and $2\sqrt{75}=10\sqrt{3}$, $\therefore$ they are like surds.

Example 6 Simplify
(a) $5\sqrt{7}-7\sqrt{7}+3\sqrt{7}$ (b) $\sqrt{18}-2\sqrt{2}+\sqrt{98}$

(a) $5\sqrt{7}-7\sqrt{7}+3\sqrt{7}=(5-7+3)\sqrt{7}=\sqrt{7}$

(b) $\sqrt{18}-2\sqrt{2}+\sqrt{98}=3\sqrt{2}-2\sqrt{2}+7\sqrt{2}=8\sqrt{2}$

The addition or subtraction of unlike surds cannot be simplified and it is left in **exact form** unless you are instructed to change it to a terminating decimal as an approximation.

Example 7 Simplify $3\sqrt{2}-\sqrt{27}+\sqrt{8}$.

$3\sqrt{2}-\sqrt{27}+\sqrt{8}=3\sqrt{2}-3\sqrt{3}+2\sqrt{2}=5\sqrt{2}-3\sqrt{3}$

Mixed surd operations

Example 8 Expand and simplify
(a) $5(2\sqrt{3}-\sqrt{2})$
(b) $\sqrt{6}(5\sqrt{3}-2)$
(c) $-\sqrt{8}(\sqrt{6}-2\sqrt{2})$

(a) $5(2\sqrt{3}-\sqrt{2})=5\times2\sqrt{3}-5\sqrt{2}=10\sqrt{3}-5\sqrt{2}$

(b) $\sqrt{6}(5\sqrt{3}-2)=5\sqrt{3}\times\sqrt{6}-2\sqrt{6}=5\sqrt{18}-2\sqrt{6}$
$=5\sqrt{9\times2}-2\sqrt{6}=15\sqrt{2}-2\sqrt{6}$

(c) $-\sqrt{8}(\sqrt{6}-2\sqrt{2})$
$=-\sqrt{48}+2\sqrt{16}=-\sqrt{16\times3}+8=-4\sqrt{3}+8$

Rationalising the denominator of an expression

Sometimes it is necessary to change the denominator of an expression to a rational number.

Example 9 Rationalise the denominator of each of the following expressions.

(a) $\frac{2}{\sqrt{7}}$ (b) $\frac{\sqrt{3}}{\sqrt{5}}$ (c) $\frac{\sqrt{2}}{2\sqrt{3}}$

(a) $\frac{2}{\sqrt{7}}=\frac{2\sqrt{7}}{\sqrt{7}\sqrt{7}}=\frac{2\sqrt{7}}{7}$

(b) $\frac{\sqrt{3}}{\sqrt{5}}=\frac{\sqrt{3}\sqrt{5}}{\sqrt{5}\sqrt{5}}=\frac{\sqrt{15}}{5}$

(c) $\frac{\sqrt{2}}{2\sqrt{3}}=\frac{\sqrt{2}\sqrt{3}}{2\sqrt{3}\sqrt{3}}=\frac{\sqrt{6}}{2\times 3}=\frac{\sqrt{6}}{6}$

Conjugate surds

$3\sqrt{5}-2\sqrt{7}$ and $3\sqrt{5}+2\sqrt{7}$ form a pair of conjugate surds. The product of a pair of conjugate surds gives a rational number.

Example 10 Expand the following pairs of conjugate surds.
(a) $(\sqrt{3}-1)(\sqrt{3}+1)$
(b) $(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})$
(c) $(3\sqrt{5}-2\sqrt{7})(3\sqrt{5}+2\sqrt{7})$

(a) $(\sqrt{3}-1)(\sqrt{3}+1)$
$=\sqrt{3}\sqrt{3}+\sqrt{3}-\sqrt{3}-1=3-1=2$

(b) $(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})$
$=\sqrt{6}\sqrt{6}-\sqrt{6}\sqrt{2}+\sqrt{2}\sqrt{6}-\sqrt{2}\sqrt{2}=6-2=4$

(c) $(3\sqrt{5}-2\sqrt{7})(3\sqrt{5}+2\sqrt{7})$
$=9\sqrt{5}\sqrt{5}+6\sqrt{5}\sqrt{7}-6\sqrt{7}\sqrt{5}-4\sqrt{7}\sqrt{7}$
$=9\times 5-4\times 7=17$

Example 11 Rationalise the denominators of the following expressions.

(a) $\frac{5}{\sqrt{3}-1}$ (b) $\frac{\sqrt{11}}{\sqrt{6}+\sqrt{2}}$ (c) $\frac{3}{3\sqrt{5}-2\sqrt{7}}$

Multiply each denominator by its conjugate surd to rationalise it.

(a) $\frac{5}{\sqrt{3}-1}=\frac{5(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}=\frac{5(\sqrt{3}+1)}{2}$

(b)
$\frac{\sqrt{11}}{\sqrt{6}+\sqrt{2}}=\frac{\sqrt{11}(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}=\frac{\sqrt{11}(\sqrt{6}-\sqrt{2})}{4}$

(c) $\frac{3}{3\sqrt{5}-2\sqrt{7}}=\frac{3(3\sqrt{5}+2\sqrt{7})}{(3\sqrt{5}-2\sqrt{7})(3\sqrt{5}+2\sqrt{7})}$
$=\frac{3(3\sqrt{5}+2\sqrt{7})}{17}$

Example 12 Calculate $8721\sqrt{3}-10681\sqrt{2}$ accurate to the 14th decimal place.
(Source: From a first year university textbook)

$8721\sqrt{3}-10681\sqrt{2}=3.3101\times 10^{-5}=0.000033101$
The difference is calculated directly with a graphics calculator, a value accurate to the 9th decimal place is obtained.

Using an indirect approach as shown below, higher accuracy can be obtained with the calculator.

$8721\sqrt{3}-10681\sqrt{2}=\frac{8721\sqrt{3}-10681\sqrt{2}}{1}$
$=\frac{(8721\sqrt{3}-10681\sqrt{2})(8721\sqrt{3}+10681\sqrt{2})}{1(8721\sqrt{3}+10681\sqrt{2})}$
$=\frac{228167523-228167522}{8721\sqrt{3}+10681\sqrt{2}}=3.310115066\times 10^{-5}$
$=0.00003310115066$

More mixed operations

Example 13 Simplify (a) $\frac{1}{\sqrt{2}}+\frac{2}{3}$ (b) $\frac{3}{\sqrt{5}}-\frac{2}{5\sqrt{3}}$

Rationalise the denominators before adding or subtracting.

(a) $\frac{1}{\sqrt{2}}+\frac{2}{3}=\frac{1\sqrt{2}}{\sqrt{2}\sqrt{2}}+\frac{2}{3}$
$=\frac{\sqrt{2}}{2}+\frac{2}{3}=\frac{3\sqrt{2}}{6}+\frac{4}{6}=\frac{3\sqrt{2}+4}{6}$

(b) $\frac{3}{\sqrt{5}}-\frac{2}{5\sqrt{3}}=\frac{3\sqrt{5}}{\sqrt{5}\sqrt{5}}-\frac{2\sqrt{3}}{5\sqrt{3}\sqrt{3}}=\frac{3\sqrt{5}}{5}-\frac{2\sqrt{3}}{15}$
$=\frac{9\sqrt{5}}{15}-\frac{2\sqrt{3}}{15}=\frac{9\sqrt{5}-2\sqrt{3}}{15}$

Example 14 Simplify $\dfrac{\sqrt{3}}{1-\sqrt{3}}-\dfrac{2\sqrt{2}}{\sqrt{2}+3}$

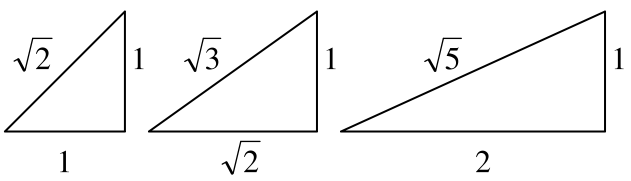
$$\frac{\sqrt{3}}{1-\sqrt{3}}-\frac{2\sqrt{2}}{\sqrt{2}+3}=\frac{\sqrt{3}\left(1+\sqrt{3}\right)}{\left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right)}-\frac{2\sqrt{2}\left(\sqrt{2}-3\right)}{\left(\sqrt{2}+3\right)\left(\sqrt{2}-3\right)}$$

$$=\frac{\sqrt{3}+3}{-2}-\frac{4-6\sqrt{2}}{-7}=\frac{-7\sqrt{3}-21}{14}+\frac{8-12\sqrt{2}}{14}$$

$$=\frac{-7\sqrt{3}-12\sqrt{2}-13}{14}$$

Locating the exact positions of $\sqrt{2},\sqrt{3},\sqrt{5}$ etc on a number line

Use Pythagoras' theorem to determine the length of $\sqrt{2},\sqrt{3},\sqrt{5}$ etc.



Start from the origin on a number line with unit scales. Draw a vertical line segment of unit length at 1 on the number line. Use the length of the line segment from the origin (as the centre) to the top end of the vertical line as the radius, draw an arc to cut across the number line. The intercept is $\sqrt{2}$.

Draw another vertical line segment of unit length at $\sqrt{2}$ on the number line. Use the length of the line segment from the origin (as the centre) to the top end of the vertical line as the radius, draw an arc to cut across the number line. The intercept is $\sqrt{3}$.

Locate $\sqrt{5}$, $\sqrt{6}$ and $\sqrt{7}$ on the number line below.

Exercise

(1) Express in standard form.
(a) $\sqrt{500}$ (b) $5\sqrt{27}$

(2) Express as entire surd. (a) $5\sqrt{7}$ (b) $^{-}8\sqrt{5}$

(3) Simplify
(a) $15\sqrt{2}-8\sqrt{2}+\sqrt{2}$ (b) $\sqrt{12}-2\sqrt{3}+\sqrt{75}$

(4) Simplify $3\sqrt{24}-\sqrt{54}+6\sqrt{8}$.

(5) Expand and simplify $3\sqrt{5}\left(2\sqrt{3}-\sqrt{20}\right)$

(6) Rationalise the denominators of the following expressions.
(a) $\dfrac{2}{\sqrt{10}}$ (b) $\dfrac{\sqrt{3}}{\sqrt{15}}$ (c) $\dfrac{6\sqrt{3}}{\sqrt{2}}$

(7) Rationalise the denominators of the following expressions.
(a) $\dfrac{1}{\sqrt{2}-1}$ (b) $\dfrac{\sqrt{27}}{\sqrt{3}+\sqrt{2}}$ (c) $\dfrac{\sqrt{3}}{3\sqrt{7}-2\sqrt{5}}$

(8) Simplify (a) $\dfrac{1}{2}+\dfrac{2}{\sqrt{3}}$ (b) $\dfrac{5}{2\sqrt{5}}-\dfrac{2}{5\sqrt{2}}$

(9) Simplify $\dfrac{\sqrt{3}}{\sqrt{3}+3}-\dfrac{2\sqrt{3}}{\sqrt{3}-3}$

(10) Simplify $\dfrac{3\sqrt{2}}{\sqrt{3}+1}-\dfrac{2\sqrt{3}}{\sqrt{2}+1}$

(11) Locate $\sqrt{10}$, $\sqrt{11}$ and $\sqrt{17}$ on a number line.