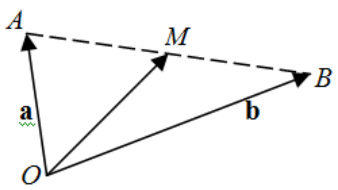
Math Lesson (Suitable for Years 11 and 12)
Proofs of concurrency of altitudes, perpendicular bisectors, medians of any triangle by vector methods

Preliminaries:

(1) Two non-zero vectors are perpendicular if their dot product equals zero.

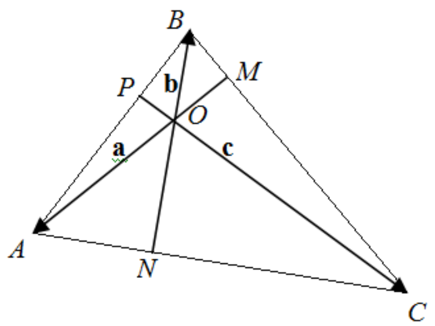
(2) If $p\mathbf{a} + q\mathbf{b} = 0$, and $\mathbf{a}$ and $\mathbf{b}$ are independent vectors, then $p = q = 0$.

(3) If M is the mid-point of $\overline{AB}$, then $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.



Concurrency of altitudes of any triangle

Proof:



$\overline{AM}$ and $\overline{BN}$ are altitudes and $\overline{CP}$ is a line segment passing through the intersection O of $\overline{AM}$ and $\overline{BN}$.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. $\therefore \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ and $\mathbf{b}.(\mathbf{c} - \mathbf{a}) = 0$,

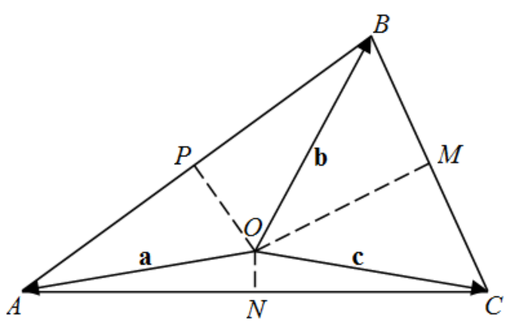
$\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$ and $\mathbf{a} \bullet (\mathbf{c} - \mathbf{b}) = 0$.

Hence $\mathbf{b}\bullet\mathbf{c} - \mathbf{b}\bullet\mathbf{a} = 0$ and $\mathbf{a}\bullet\mathbf{c} - \mathbf{a}\bullet\mathbf{b} = 0$,

$\therefore \mathbf{b}\bullet\mathbf{c} - \mathbf{a}\bullet\mathbf{c} = 0$, or $(\mathbf{b} - \mathbf{a})\bullet\mathbf{c} = 0$, i.e. $\mathbf{c}$ is perpendicular to $\overrightarrow{AB}$ and hence $\overline{CP}$ is perpendicular to $\overline{AB}$, i.e. $\overline{CP}$ is an altitude. Hence the three altitudes are concurrent. A similar proof can be constructed for an obtuse triangle. Try it as an exercise.

Concurrency of perpendicular bisectors of any triangle

Proof:



$\overline{OM}$ and $\overline{ON}$ are perpendicular bisectors of $\overline{BC}$ and $\overline{AC}$ respectively, and $\overline{OP}$ bisects $\overline{AB}$.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. $\therefore \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$, $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$ and $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$.

Since M, N and P are mid-points,

$\therefore \overrightarrow{OM} = \frac{1}{2}(\mathbf{c} + \mathbf{b})$, $\overrightarrow{ON} = \frac{1}{2}(\mathbf{c} + \mathbf{a})$ and $\overrightarrow{OP} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Since $\overline{OM}$ and $\overline{ON}$ are perpendicular to $\overline{BC}$ and $\overline{AC}$ respectively, $\therefore \overrightarrow{OM} \bullet \overrightarrow{BC} = \frac{1}{2}(\mathbf{c} + \mathbf{b})\bullet(\mathbf{c} - \mathbf{b}) = 0$ and

$\overrightarrow{ON} \bullet \overrightarrow{AC} = \frac{1}{2}(\mathbf{c} + \mathbf{a})\bullet(\mathbf{c} - \mathbf{a}) = 0$.

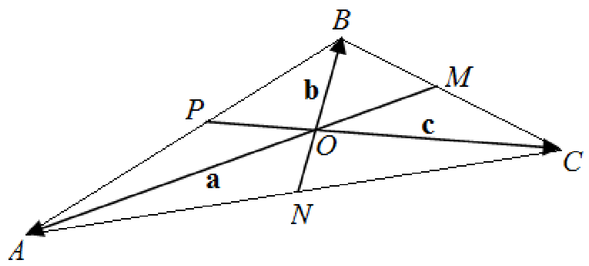
Hence $|\mathbf{c}|^2 - |\mathbf{b}|^2 = 0$ and $|\mathbf{c}|^2 - |\mathbf{a}|^2 = 0$, $\therefore |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$.

$\overrightarrow{OP} \bullet \overrightarrow{BA} = \frac{1}{2}(\mathbf{a} + \mathbf{b}).(\mathbf{a} - \mathbf{b}) = \frac{1}{2}\left[\, |\mathbf{a}|^2 - |\mathbf{b}|^2 \,\right] = 0$,

$\therefore \overline{OP}$ is perpendicular to $\overline{BA}$, $\therefore$ it is a perpendicular bisector of $\overline{BA}$. Hence the three perpendicular bisectors are concurrent.

Concurrency of medians of any triangle

Proof:



$\overline{AM}$ and $\overline{BN}$ are medians and $\overline{CP}$ is a line segment passing through the intersection O of $\overline{AM}$ and $\overline{BN}$.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

$\therefore \overrightarrow{OM} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $\overrightarrow{ON} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$.

Let m, n and p be some positive constants such that $\overrightarrow{OM} = -m\,\mathbf{a}$, $\overrightarrow{ON} = -n\,\mathbf{b}$ and $\overrightarrow{OP} = -p\,\mathbf{c}$.

$\therefore -m\,\mathbf{a} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $-n\,\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$.

From the last two equations, $-2m\mathbf{a} - \mathbf{b} = -2n\mathbf{b} - \mathbf{a}$,

hence $(1 - 2m)\mathbf{a} - (1 - 2n)\mathbf{b} = 0$.

Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, $\therefore$ they are independent and hence

$1 - 2m = 0$ and $-(1 - 2n) = 0$, i.e. $m = n = \frac{1}{2}$.

$\therefore -\frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$ and $\mathbf{b} = -\mathbf{a} - \mathbf{c}$.

$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\mathbf{a} - p\mathbf{c}$, and $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = -2\mathbf{a} - \mathbf{c}$.

Let $\overrightarrow{AP} = k\overrightarrow{AB}$ where k is a positive constant.

$\therefore -\mathbf{a} - p\mathbf{c} = k(-2\mathbf{a} - \mathbf{c})$ or $(2k - 1)\mathbf{a} + (k - p)\mathbf{c} = 0$.

Since $\mathbf{a}$ and $\mathbf{c}$ are independent, $\therefore 2k - 1 = 0$ and $k - p = 0$,

i.e. $k = \frac{1}{2}$ and $p = \frac{1}{2}$.

$\therefore P$ is the mid-point of $\overline{AB}$ and $\overline{CP}$ is a median.

Hence the three medians are concurrent and the intersection trisects each one.