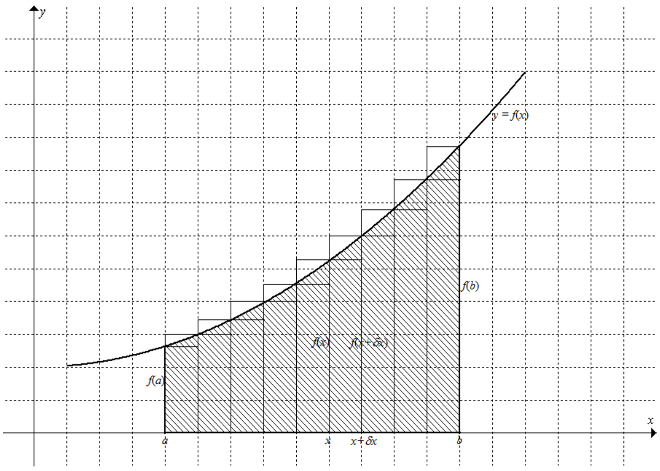
Lesson – Connection between the antiderivative of a function and the area under the graph

Area of the region bounded by the graph of a function and the *x*-axis



The shaded region can be approximated by a series of lower rectangles or upper rectangles.

The area measure of the shaded region *A* lies between the sum of the area measures of the lower rectangles S_L and the sum of the area measures of the upper rectangles S_U.

$$S_L < A < S_U$$

$$\sum_a^b f(x)\delta x < A < \sum_a^b f(x+\delta x)\delta x$$

As the region between $x=a$ and $x=b$ is divided into more and smaller lower and upper rectangles, $\delta x \to 0$, $S_U \to S_L$ and $\lim_{\delta x\to 0}\sum_a^b f(x)\delta x = A$.

$\lim_{\delta x\to 0}\sum_a^b f(x)\delta x$, denoted by Leibniz as $\int_a^b f(x)dx$ and called the **integral** of $f(x)$ from *a* to *b*, gives the area of the shaded region *A*.

Relationship between the antiderivative of $f(x)$ and the area under its graph

Let $A(x)$ be the area of the region under the curve from *a* to x, and $A(x+\delta x)$ the area of the region from *a* to $x+\delta x$.

.: the area of the region from x to $x+\delta x$ is $A(x+\delta x)-A(x)$

.: $f(x)\delta x < A(x+\delta x)-A(x) < f(x+\delta x)\delta x$ where $\delta x \neq 0$

.: $f(x) < \dfrac{A(x+\delta x)-A(x)}{\delta x} < f(x+\delta x)$

As $\delta x \to 0$,

$f(x+\delta x)\to f(x)$ and $\dfrac{A(x+\delta x)-A(x)}{\delta x} \to \dfrac{d}{dx}A(x)$

.: $f(x)=\dfrac{d}{dx}A(x)$.

Let $F(x)$ be an antiderivative of $f(x)$, .: $A(x)=F(x)+c$ where *c* is a real constant.

When $x=a$, $A(a)=F(a)+c=0$, .: $c=-F(a)$

When $x=b$, $A(b)=F(b)+c=F(b)-F(a)$

Since $A(b)=A$, .: $\int_a^b f(x)dx = F(b)-F(a)$ and it is called the **fundamental theorem of calculus**. The theorem is true for any function continuous in the interval $[a,b]$.

In the evaluation of the integral $\int_a^b f(x)dx$, antidifferentiate $f(x)$ to obtain $F(x)$ first, then calculate $F(b)-F(a)$.

.: it is convenient to introduce an intermediate step in the evaluation.

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a)$$

Example 1 Given the antiderivative of $f(x)$ is $\sqrt{x}$, find the area of the region bounded by the curve $y=f(x)$, the *x*-axis, $x=\frac{1}{4}$ and $x=9$.

$F(x)=\sqrt{x}$

$A=\int_{\frac{1}{4}}^{9} f(x)dx = F(9)-F\left(\frac{1}{4}\right)$

$=\sqrt{9}-\sqrt{\frac{1}{4}}=3-\frac{1}{2}=\frac{5}{2}$

Example 2 Find the area of the region bounded by the curve $y=e^{\frac{x}{2}}$, the *x*-axis, $x=0$ and $x=\log_e 9$.

$$A=\int_0^{\log_e 9} e^{\frac{x}{2}}dx=\left[2e^{\frac{x}{2}}\right]_0^{\log_e 9}=2e^{\frac{1}{2}\log_e 9}-2e^0=4$$

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