2003 VCAA Mathematical Methods Exam 2 Solutions
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Q1a $\Pr(X > 141.5) = 0.106$ (graphics calc.)

Q1b $\Pr(X < 140 - d) = \frac{0.15}{2} = 0.075$

$\Pr\left(Z < \frac{140 - d - 140}{1.2}\right) = 0.075$

$\frac{-d}{1.2} = -1.440$ (graphics calc.)

$d = 1.7$

Q1c Use binomial as an approximation, $n = 2$, $p = 0.15$. $\Pr(X = 2) = {}^{12}C_2 (0.15)^2 (0.85)^{10} = 0.292$.

Q1d Hypergeometric,

$\Pr(X \geq 2) = 1 - \Pr(X = 0) - \Pr(X = 1)$

$= 1 - \frac{{}^4C_0 \times {}^{21}C_{12}}{{}^{25}C_{12}} - \frac{{}^4C_1 \times {}^{21}C_{11}}{{}^{25}C_{12}}$

$= 0.672$

Q1ei $k + 0.15 + 0.17 = 1$, $k = 0.68$

Q1eii $\mu = E(Y) = 0.68(x - 5) + 0.15 \times 0 + 0.17(x - 8)$

Q1eiii $0.68(x - 5) + 0.15 \times 0 + 0.17(x - 8) = 0$

$x = 5.60$ dollars

Q1eiv $\frac{k}{k + 0.17} = \frac{0.68}{0.85} = 0.8$

Q2a At $x = 4$, max. height $= 2 - 2\cos\left(\frac{4}{2}\right) = 2.83$

Q2b $\frac{dy}{dx} = \sin\left(\frac{x}{2}\right)$. Since $0 \leq \left|\sin\left(\frac{x}{2}\right)\right| \leq 1$,

$\therefore \left|\frac{dy}{dx}\right| \leq 1$.

Q2c Area $= 2 \times \int_0^4 \left(2 - 2\cos\left(\frac{x}{2}\right)\right) dx$

$= 2 \times \left[2x - 4\sin\left(\frac{x}{2}\right)\right]_0^4$

$= 2 \times \left[\left(2 \times 4 - 4\sin\left(\frac{4}{2}\right)\right) - (0)\right] = 8.73\ m^2$.

Q2di At $y = 1$, $1 = 2 - 2\cos\left(\frac{x}{2}\right)$, $\cos\left(\frac{x}{2}\right) = \frac{1}{2}$,

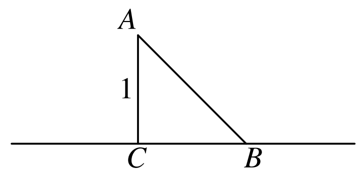
$\frac{x}{2} = \frac{\pi}{3}$, $\therefore x = \frac{2\pi}{3}$

Q2dii Gradient of tangent at A $= \frac{dy}{dx} = \sin\left(\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2}$.

Gradient of normal at A $= -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$.

Q2diii



Gradient of $AB = -\frac{2}{\sqrt{3}}$, $\therefore \frac{1}{\overline{CB}} = \frac{2}{\sqrt{3}}$, $\overline{CB} = \frac{\sqrt{3}}{2}$.

$\therefore \overline{AB} = \sqrt{1^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\sqrt{7}}{2}$

Q3a $f'(x) = x^3\left(-2e^{-2x}\right) + \left(3x^2\right)e^{-2x}$

$= e^{-2x}\left(-2x^3 + 3x^2\right)$

$\therefore a = -2$ and $b = 3$.

Q3b At stationary points, $f'(x) = 0$.

$\therefore e^{-2x}\left(-2x^3 + 3x^2\right) = 0$. Since $e^{-2x} \neq 0$,

$-2x^3 + 3x^2 = 0$, $x^2(-2x + 3) = 0$, $x = 0$ or $x = \frac{3}{2}$.

The corresponding y values are 0 and $\frac{27}{8e^3}$.

$(0,0)$ is an inflection point and $\left(\frac{3}{2}, \frac{27}{8e^3}\right)$ a local maximum.

Q3ci At $x = 1$, $y = f(1) = \frac{1}{e^2}$,

and gradient of tangent $= f'(1) = \frac{1}{e^2}$.

Equation of tangent is $y - \frac{1}{e^2} = \frac{1}{e^2}(x - 1)$, i.e.

$y = \frac{1}{e^2}x$.

Q3cii At $x=0$, gradient of tangent = 0 (stationary point), equation of tangent is $y=0$.

Q3ciii Let (x_1, y_1) be a point on the curve, $\therefore y_1 = x_1^3 e^{-2x_1}$.

Equation of tangent at (x_1, y_1) is

$y - y_1 = e^{-2x_1}\left(-2x_1^3 + 3x_1^2\right)(x - x_1)$.

$\therefore y = e^{-2x_1}\left(-2x_1^3 + 3x_1^2\right)(x - x_1) + y_1$

$\therefore y = e^{-2x_1}\left(-2x_1^3 + 3x_1^2\right)x - e^{-2x_1}\left(-2x_1^3 + 3x_1^2\right)x_1 + x_1^3 e^{-2x_1}$

Tangents pass through the origin, y-intercept = 0,

$-e^{-2x_1}\left(-2x_1^3 + 3x_1^2\right)x_1 + x_1^3 e^{-2x_1} = 0$,

$-e^{-2x_1}\left[\left(-2x_1^3 + 3x_1^2\right)x_1 - x_1^3\right] = 0$,

$\therefore -2x_1^4 + 2x_1^3 = 0$,

$2x_1^3(-x_1 + 1) = 0$, $\therefore x_1 = 0$ or $x_1 = 1$ are the only two possibilities.

Q3di

$(12x^2 + 2px + q)e^{-2x} - (4x^3 + px^2 + qx + 3)(2e^{-2x}) = kx^3 e^{-2x}$

$12x^2 + 2px + q - 8x^3 - 2px^2 - 2qx - 6 = kx^3$

$-8x^3 + (12 - 2p)x^2 + (2p - 2q)x + (q - 6) = kx^3$

$\therefore k = -8$, $p = q = 6$

Q3dii Area under $y = f(x)$ between $x=0$ and 1

$= \int_0^1 x^3 e^{-2x}\,dx = \frac{1}{k}\left[(4x^3 + px^2 + qx + 3)e^{-2x}\right]_0^1$

$= \frac{1}{-8}\left[(4x^3 + 6x^2 + 6x + 3)e^{-2x}\right]_0^1 = \frac{1}{-8}\left[19e^{-2} - 3\right]$

$= \frac{3 - 19e^{-2}}{8}$.

Area under tangent (between $x=0$ and 1)

$= \frac{1}{2} \times 1 \times e^{-2} = \frac{e^{-2}}{2}$.

$\therefore$ Area between $y = f(x)$ and tangent is

$\frac{e^{-2}}{2} - \frac{3 - 19e^{-2}}{8} = \frac{23e^{-2} - 3}{8}$.

Q4a $\frac{3t}{5 + t^2} = 0.4$, $0.4t^2 + 2 = 3t$, $0.4t^2 - 3t + 2 = 0$,

$t = 0.7396$ or 6.7604.

Length of time $= 6.7604 - 0.7396 = 6.02$

Q4b Max concentration when $\frac{dx}{dt} = 0$,

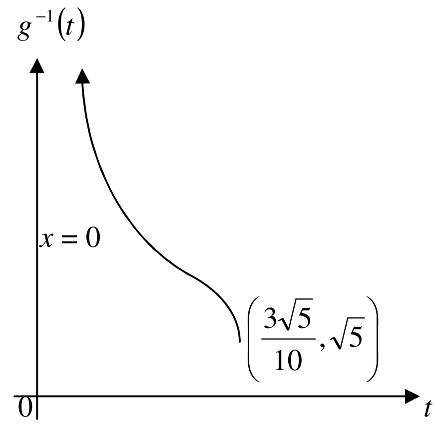
i.e. $\frac{3(5 + t^2) - 3t(2t)}{(5 + t^2)^2} = 0$, $\therefore 15 - 3t^2 = 0$, $t = \sqrt{5}$.

Max concentration $= \frac{3\sqrt{5}}{10}$ mg/L.

Q4c $\frac{15 - 3t^2}{(5 + t^2)^2} = 0.25$, $t = 1.22$ (graphics calc.)

Q4di $g(t)$ is a one-to-one function when $a = \sqrt{5}$ and $g^{-1}(t)$ exists.

Q4dii



Qdiii Equation of function g: $x = \frac{3t}{5 + t^2}$

Equation of function g^{-1}: $t = \frac{3x}{5 + x^2}$

$t(5 + x^2) = 3x$, $tx^2 - 3x + 5t = 0$,

$\therefore x = \frac{3 \pm \sqrt{9 - 20t^2}}{2t}$ (quadratic formula)

$\therefore g^{-1}(t) = \frac{3 + \sqrt{9 - 20t^2}}{2t}$ for $0 < t \le \frac{3\sqrt{5}}{10}$.

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