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2005 NSW BOS Mathematics Extension 1 Solutions
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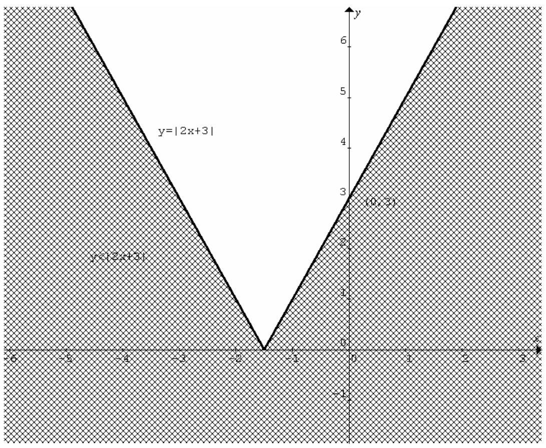
Q1a $\int \frac{1}{x^2+49}dx = \int \frac{1}{7^2+x^2}dx = \frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)+C$.

Q1b



Q1c For $y=\cos^{-1}\left(\frac{x}{4}\right)$, the domain is $[-4,4]$, the range $[0,\pi]$.

Q1d Let $u=2x^2+1$, $\frac{du}{dx}=4x$, or $x=\frac{1}{4}\frac{du}{dx}$.

$\therefore \int x\left(2x^2+1\right)^{\frac{5}{4}}dx = \int \frac{1}{4}u^{\frac{5}{4}}\frac{du}{dx}dx = \int \frac{1}{4}u^{\frac{5}{4}}du$

$= \frac{1}{9}u^{\frac{9}{4}}+C = \frac{1}{9}\left(2x^2+1\right)^{\frac{9}{4}}+C.$

Q1e $\frac{3\times^- 1+2x}{5}=1$ and $\frac{3\times 8+2y}{5}=4$, $\therefore x=4$ and $y=-2$.

Q1f Let $\tan\theta=3$ and $\tan\phi=m$, $\therefore$ either $\theta-\phi=45^o$ or $\phi-\theta=45^o$. Hence $\tan(\theta-\phi)=1$ or $\tan(\phi-\theta)=1$.

$\therefore \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{3-m}{1+3m}=1$ or $\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}=\frac{m-3}{1+3m}=1$.

$\therefore m=\frac{1}{2}$ or -2 .

Q2a $\frac{d}{dx}\left(2\sin^{-1}(5x)\right)=5\times\frac{2}{\sqrt{1-(5x)^2}}=\frac{10}{\sqrt{1-25x^2}}$.

Q2b The term independent of x in the expansion of $\left(2x-\frac{1}{x^2}\right)^{12}$ is ${}^{12}C_4(2x)^8\left(-\frac{1}{x^2}\right)^4=126720$.

Q2ci Apply the product rule to find $\frac{d}{dx}\left(e^{3x}(\cos x-3\sin x)\right)$

$=e^{3x}(-\sin x-3\cos x)+3e^{3x}(\cos x-3\sin x)$

$=-10e^{3x}\sin x$.

Q2cii $\int -10e^{3x}\sin x dx = e^{3x}(\cos x-3\sin x)+D$,

$\therefore \int e^{3x}\sin x dx = -\frac{1}{10}e^{3x}(\cos x-3\sin x)+C$.

Q2di $T=3+Ae^{-kt}$, $\therefore \frac{dT}{dt}=-kAe^{-kt}$ and $-k(T-3)=-kAe^{-kt}$.

$\therefore T=3+Ae^{-kt}$ satisfies the differential equation.

Q2dii At $t=0$, $T=25$. At $t=10$, $T=11$. Substitute into $T=3+Ae^{-kt}$ to obtain $25=3+A$, $\therefore A=22$ and $11=3+Ae^{-10k}$, i.e. $\frac{4}{11}=e^{-10k}$, or $k=\frac{1}{10}\log_e\left(\frac{11}{4}\right)$.

At $t=15$, $T=3+22e^{-15k}=7.8^o$ C.

Q3ai $g(0.7)=-0.041$, $g(0.9)=0.168$. $\therefore g(x)$ has a zero between 0.7 and 0.9.

Q3aii Halving the interval, $x=\frac{0.7+0.9}{2}=0.8$,

$g(0.8)=0.052$, $\therefore g(x)$ has a zero between 0.7 and 0.8.

Halving the interval, $x=\frac{0.7+0.8}{2}=0.75$,

$g(0.75)=0.003$, $\therefore g(x)$ has a zero between 0.7 and 0.75.

To one decimal place, $\therefore g(x)$ has a zero at 0.7.

Q3bi $\sin(5x+4x)+\sin(5x-4x)$

$=\sin 5x\cos 4x+\cos 5x\sin 4x+\sin 5x\cos 4x-\cos 5x\sin 4x$

$=2\sin 5x\cos 4x$.

Q3bii $\int \sin 5x\cos 4x dx = \int \frac{1}{2}(\sin 9x+\sin x)dx$

$=-\frac{1}{18}\cos 9x-\frac{1}{2}\cos x+C$.

Q3c $f(x)=x^2+5x$, $f(x+h)=(x+h)^2+5(x+h)$,

$f'(x)=\lim_{h\to 0}\frac{(x+h)^2+5(x+h)-x^2-5x}{h}=\lim_{h\to 0}(2x+h+5)=2x+5$

Q3di $EB=7-4=3$, $EC=\ell-x$. $AE\times EB=DE\times EC$,

$\therefore 12=x(\ell-x)$, or $x^2-\ell x+12=0$.

Q3dii For this quadratic equation to have real x for its solutions, $\Delta\geq 0$, i.e. $\ell^2-4(1)(12)\geq 0$, $\ell^2\geq 48$. Since $\ell>0$, $\therefore \ell\geq\sqrt{48}$. Hence the shortest chord has length $\sqrt{48}=4\sqrt{3}$.

Q4a Let $u = \sin x$, when $x = 0$, $u = 0$; when $x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$.

$\frac{du}{dx} = \cos x$.

$$\int_0^{\frac{\pi}{4}} \cos x \sin^2 x dx = \int_0^{\frac{1}{\sqrt{2}}} u^2 \frac{du}{dx} dx = \int_0^{\frac{1}{\sqrt{2}}} u^2 du = \left[\frac{u^3}{3}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{12}.$$

Q4b $\cos ec\theta + \cot\theta = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \frac{1+\cos\theta}{\sin\theta}$

$$= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}.$$

Q4ci Solve simultaneously to find R,

$x + py = 2ap + ap^3$ …….(1)

$x + qy = 2aq + aq^3$ …….(2) where $p \neq q$.

(2) – (1), $qy - py = 2aq - 2ap + aq^3 - ap^3$,

$(q-p)y = 2a(q-p) + a(q^3 - p^3)$,

$(q-p)y = 2a(q-p) + a(q-p)(q^2 + qp + p^2)$,

$(q-p)y = a(q-p)(2 + q^2 + qp + p^2)$.

Hence $y = a(p^2 + pq + q^2 + 2)$……(3)

Substitute (3) into (1), $x = 2ap + ap^3 - py$

$= 2ap + ap^3 - pa(p^2 + pq + q^2 + 2) = -apq(p+q)$.

$\therefore R$ is $(-apq[p+q], a[p^2 + pq + q^2 + 2])$.

Q4cii $(0, a)$ satisfies $y = \frac{1}{2}(p+q)x - apq$, $\therefore a = -apq$ and $pq = -1$.

Q4ciii Since $pq = -1$, $\therefore R$ is $(a[p+q], a[p^2 + pq + q^2 + 2])$.

$\therefore x = a(p+q)$ or $\frac{x}{a} = (p+q)$…….(1) and

$y = a(p^2 + pq + q^2 + 2) = a(p^2 + 2pq + q^2 - pq + 2)$

$= a((p+q)^2 + 1 + 2)$,

i.e. $y = a((p+q)^2 + 3)$………(2)

Substitute (1) into (2), $y = a\left(\left(\frac{x}{a}\right)^2 + 3\right)$, $\therefore y = \frac{1}{a}x^2 + 3a$.

Q4d For $n = 2$, $4^n - 1 - 7n = 4^2 - 1 - 7 \times 2 = 1 > 0$.

Assume $4^n - 1 - 7n > 0$ is true for $n = k > 2$,

i.e. $4^k - 1 - 7k > 0$, then for $n = k + 1$,

$4^{k+1} - 1 - 7(k+1) = 4 \times 4^k - 1 - 7k - 7$

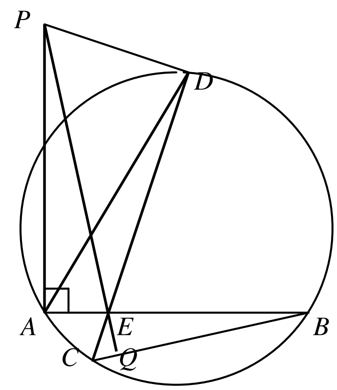
$= 4(4^k - 1 - 7k) + 21k - 4 > 0$. Hence $4^n - 1 - 7n > 0$ for all $n \geq 2$.

Q5a $V = \int_0^{\frac{\pi}{8}} \pi \sin^2 2x dx = \int_0^{\frac{\pi}{8}} \frac{\pi}{2}(1 - \cos 4x) dx = \left[\frac{\pi}{2}\left(x - \frac{\sin 4x}{4}\right)\right]_0^{\frac{\pi}{8}}$

$= \frac{\pi}{2}\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi(\pi - 2)}{16}$.

Q5bi Quadrilateral $DPAE$ is cyclic because the sum of the opposite angles is 180^o.

Q5bii $\angle PDC = 90^o$



$\angle APE = \angle ADE$, because they are subtended by the same arc AE of the circle that passes through the vertices D, P, A and E. The circle is not shown in the above diagram.

$\angle ADC = \angle ABC$, because they are subtended by the same arc AC of the circle shown above. Since $\angle ADE$ and $\angle ADC$ are the same angle, $\therefore \angle APE = \angle ABC$.

Q5biii Consider ΔAPE and ΔQBE.

Since $\angle APE = \angle ABC = \angle QBE$ and $\angle AEP = \angle QEB$ (vertically opposite angles are equal), $\therefore$ the third pair of angles must be the same, i.e. $\angle EQB = \angle EAP = 90^o$. $\therefore PQ \perp BC$.

Q5ci Let $\sqrt{3}\sin 3t - \cos 3t = R\sin(3t - \alpha)$, re-express the RHS to obtain $\sqrt{3}\sin 3t - \cos 3t = R\sin 3t \cos\alpha - R\cos 3t \sin\alpha$.

Compare the two sides, $R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$.

Hence $\tan\alpha = \frac{1}{\sqrt{3}}$ and $R^2\cos^2\alpha + R^2\sin^2\alpha = 4$.

$\therefore \alpha = \frac{\pi}{6}$ and $R = 2$. $\therefore \sqrt{3}\sin 3t - \cos 3t = 2\sin\left(3t - \frac{\pi}{6}\right)$.

Q5cii $x = 5 + \sqrt{3}\sin 3t - \cos 3t = 5 + 2\sin\left(3t - \frac{\pi}{6}\right)$, hence the particle oscillates about $x = 5$, the centre of motion, with an amplitude of 2 units.

Q5ciii Maximum speed occurs when the particle passes through the centre of motion, where $\sin\left(3t - \frac{\pi}{6}\right) = 0$,

$3t - \frac{\pi}{6} = 0$, $t = \frac{\pi}{18}$.

Q6ai Binomial distribution: $n = 5,\ p = \frac{2}{3}$.

To earn one point Megan needs to pick 3 or more.

$$\Pr(X \ge 3) = {}^5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 + {}^5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1 + {}^5C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0 = 0.7901$$

Q6aii Binomial distribution: $n = 18,\ p = 0.7901$.

$\Pr(X = 18) = 0.7901^{18} = 0.01$.

Q6aiii Binomial distribution: $n = 18,\ p = 0.7901,\ q = 0.2099$.

$\Pr(X \le 16) = 1 - \Pr(X = 17) - \Pr(X = 18)$

$= 1 - {}^{18}C_{17}(0.7901)^{17}(0.2099)^1 - 0.01 = 0.92$.

Q6bi Maximum height is reached when $\frac{dy}{dt} = 0$.

$y = -4.9t^2 + 200t + 5000,\ \frac{dy}{dt} = -9.8t + 200 = 0,\ \therefore t = 20.4$ s and $y = 7040.8$ m.

Q6bii $x = 200t,\ \therefore \frac{dx}{dt} = 200$.

Descending slope is $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{-9.8t + 200}{200}$.

At 45^o, $\frac{dy}{dx} = \tan(-45^o) = -1,\ \therefore \frac{-9.8t + 200}{200} = -1,\ t = 40.8$ s.

At 60^o, $\frac{dy}{dx} = \tan(-60^o) = -\sqrt{3}$,

$\therefore \frac{-9.8t + 200}{200} = -\sqrt{3},\ t = 55.8$ s.

Earliest time is 40.8 s and the latest time is 55.8 s.

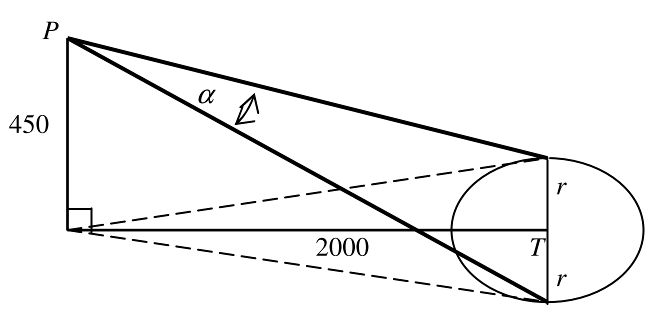
Q6biii The latest time is when the speed = 350 ms^{-1},

i.e. $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 350,\ \therefore 200^2 + (200 - 9.8t)^2 = 122500$,

$t = 49.7$ s.

Q7ai $PT = \sqrt{450^2 + 2000^2} = 2050$ m

$\frac{r}{2050} = \tan(0.05),\ \therefore r = 102.6$ m.



Q7aii $r = 2050\tan\left(\frac{\alpha}{2}\right),\ \therefore \frac{dr}{d\alpha} = 1025\sec^2\left(\frac{\alpha}{2}\right)$.

Given $\frac{d\alpha}{dt} = 0.02$ and $\alpha = 0.1$,

$\therefore \frac{dr}{dt} = \frac{dr}{d\alpha} \times \frac{d\alpha}{dt} = 20.5\sec^2(0.05) = 20.6$ m per hour.

Q7bi $f(x) = Ax^3 - Ax + 1$, where $A > 0$.

$f'(x) = 3Ax^2 - A.\ \therefore f'\left(\pm\frac{\sqrt{3}}{3}\right) = 3A\left(\pm\frac{\sqrt{3}}{3}\right)^2 - A = 0.$

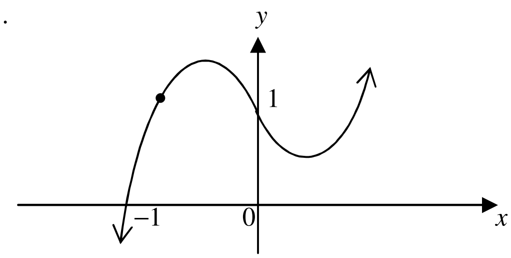
$\therefore f(x)$ has stationary points at $x = \pm\frac{\sqrt{3}}{3}$.

Q7bii At $x = \frac{\sqrt{3}}{3}$, the value of the function is a minimum,

$y = f\left(\frac{\sqrt{3}}{3}\right) = A\left(\frac{\sqrt{3}}{3}\right)^3 - A\left(\frac{\sqrt{3}}{3}\right) + 1 = -\frac{2A}{3\sqrt{3}} + 1.$

For the local minimum to be a positive value, $-\frac{2A}{3\sqrt{3}} + 1 > 0$,

$\therefore A < \frac{3\sqrt{3}}{2}$.



$\therefore f(x)$ has exactly one zero when $A < \frac{3\sqrt{3}}{2}$.

Q7biii Since $f(-1) = 1$ and $f'(-1) = 2A > 0$, given $A > 0$, the only zero must be at $x < -1$. $\therefore f(x)$ does not have a zero in the interval $-1 \le x \le 1$ when $0 < A < \frac{3\sqrt{3}}{2}$.

Q7biv $g(\theta) = 2\cos\theta + \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$g'(\theta) = -2\sin\theta + \sec^2\theta = -2\sin\theta + \frac{1}{\cos^2\theta}$

$= \frac{-2\sin\theta\cos^2\theta + 1}{\cos^2\theta} = \frac{-2\sin\theta(1 - \sin^2\theta) + 1}{\cos^2\theta}$

$= \frac{2\sin^3\theta - 2\sin\theta + 1}{\cos^2\theta}$. Since $0 < 2 < \frac{3\sqrt{3}}{2}$ and $-1 < \sin\theta < 1$

$\because -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ \therefore 2\sin^3\theta - 2\sin\theta + 1$ has no zeros. Hence $g(\theta)$ does not have any stationary points.

Q7bv $\therefore g(\theta)$ must be a one-to-one function in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\therefore g(\theta)$ has an inverse function.

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