2005 NSW BOS Mathematics Extension 2 Solutions
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Q1a Let $u=\sin\theta,\ \frac{du}{d\theta}=\cos\theta.\ \int\frac{\cos\theta}{\sin^5\theta}d\theta=\int\frac{1}{u^5}\frac{du}{d\theta}d\theta$

$=\int u^{-5}du=\frac{u^{-4}}{-4}+C=-\frac{1}{4\sin^4\theta}+C.$

Q1bi $5x\equiv a(x+2)+b(x-3)$. Let $x=-2$, then $b=2$. Let $x=3$, then $a=3$.

Q1bii
$\int\frac{5x}{x^2-x-6}dx=\int\frac{3}{x-3}+\frac{2}{x+2}dx=3\log_e(x-3)+2\log_e(x+2)+C$
$=\log_e(x-3)^3(x+2)^2+C.$

Q1c
$\int_1^e x^7\log_e x\,dx=\int_1^e(\log_e x)x^7dx=\left[(\log_e x)\left(\frac{x^8}{8}\right)\right]_1^e-\int_1^e\left(\frac{1}{x}\right)\left(\frac{x^8}{8}\right)dx$
$=\frac{e^8}{8}-\left[\frac{x^8}{64}\right]_1^e=\frac{e^8}{8}-\frac{e^8}{64}+\frac{1}{64}=\frac{7e^8+1}{64}.$

Q1d $\int\frac{dx}{\sqrt{(2x)^2-1}}=\frac{1}{2}\log_e\left(2x+\sqrt{(2x)^2-1}\right)+C$
$=\frac{1}{2}\log_e\left(2x+\sqrt{4x^2-1}\right)+C$.

Q1ei $t=\tan\frac{\theta}{2},\ \frac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{\theta}{2}=\frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)=\frac{1}{2}\left(1+t^2\right)$.

Q1eii $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}=\frac{2t}{1+t^2}$.

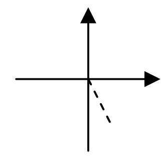
Q1eiii $\int\cos ec\theta d\theta=\int\frac{1}{\sin\theta}d\theta=\int\frac{1+t^2}{2t}d\theta=\int\frac{1}{t}\frac{dt}{d\theta}d\theta$
$=\int\frac{1}{t}dt=\log_e t+C=\log_e\left(\tan\frac{\theta}{2}\right)+C.$

Q2ai $2z+iw=2(3+i)+i(1-i)=7+3i$.

Q2aii $\bar{z}w=(3-i)(1-i)=2-4i$.

Q2aiii $\frac{6}{w}=\frac{6\bar{w}}{w\bar{w}}=\frac{6(1+i)}{2}=3(1+i)$.

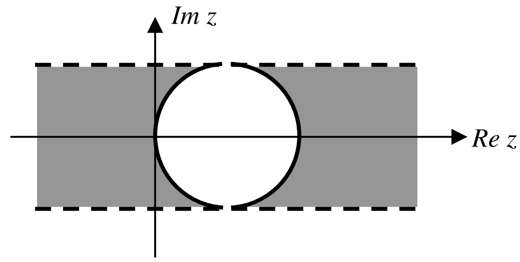
Q2bi $\beta=1-i\sqrt{3}$, $r=\sqrt{1^2+\left(-\sqrt{3}\right)^2}=2$,
$\theta=\tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)=-\frac{\pi}{3}.\ \therefore\beta=2cis\left(-\frac{\pi}{3}\right).$



Q2bii $\beta^5=2^5cis5\left(-\frac{\pi}{3}\right)=32cis\left(-\frac{5\pi}{3}\right)=32cis\left(\frac{\pi}{3}\right)$.

Q2biii $\beta^5=32\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=16+\left(16\sqrt{3}\right)i$.

Q2c Let $z=x+iy$,
$\bar{z}=x-iy,\ z-\bar{z}=i2y,\ |z-\bar{z}|=\sqrt{4y^2}=2|y|<2,\ \therefore|y|<1$,
i.e. all the complex numbers with imaginary part $-1<y<1$.
$|z-1|\geq1$ consists of all the complex numbers on or outside the circle with radius 1 unit and centred at 1+0*i*.



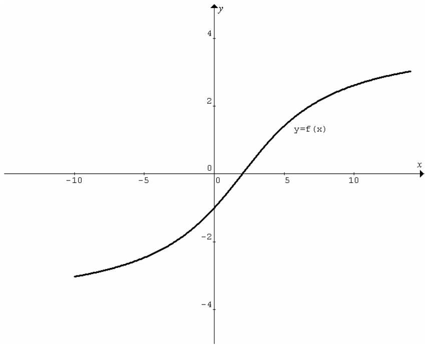
Q2di Let $\theta=\arg(z_1)$. Then the angle between ℓ and *OP* is $(\alpha-\theta)$. $\therefore\arg(z_2)=\alpha+(\alpha-\theta)=2\alpha-\theta$.
$\therefore\arg(z_1)+\arg(z_2)=\theta+2\alpha-\theta=2\alpha$.

Q2dii Since $|z_2|=|z_1|$ and $\arg(z_1)+\arg(z_2)=2\alpha$,
$\therefore z_1z_2=|z_1||z_2|cis(\arg(z_1)+\arg(z_2))=|z_1|^2cis(2\alpha)$.

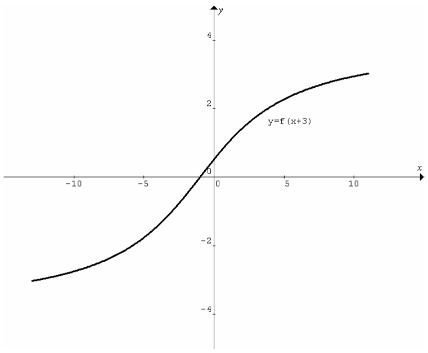
Q2diii When $\alpha=\frac{\pi}{4},\ z_1z_2=|z_1|^2cis\left(\frac{\pi}{2}\right)=i|z_1|^2$.

Since $|z_1|>0$, $\therefore$ the complex number z_1z_2 is a point *R* on the positive imaginary axis. As $|z_1|$ increases, *R* moves in the positive direction. The argument of z_1 has no effects on the locus of *R*.

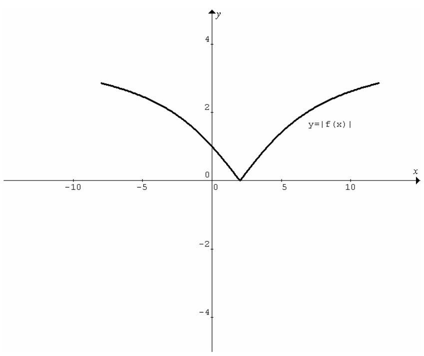
Q3a Given the graph of $y = f(x)$.



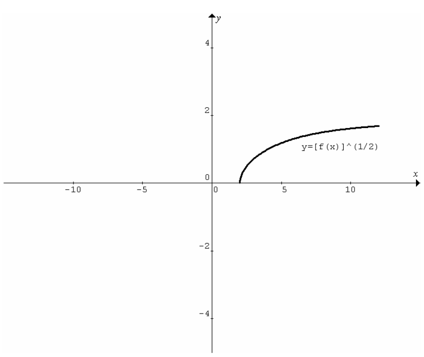
Q3ai



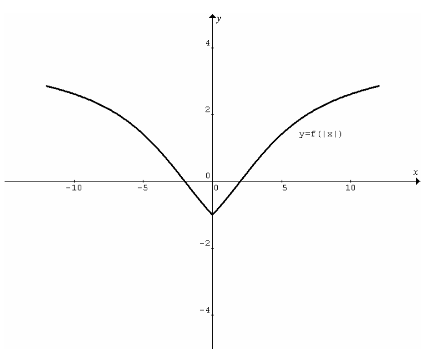
Q3aii



Q3aiii



Q3aiv

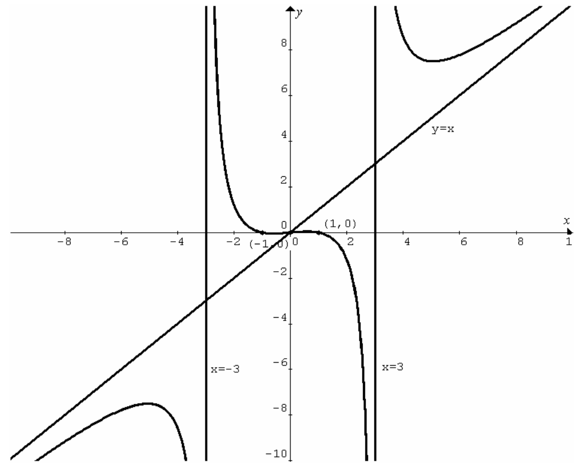


Q3b $y = x + \frac{8x}{x^2 - 9} = x + \frac{8x}{(x-3)(x+3)}$.

Asymptotes are $y = x,\ x = 3,\ x = -3$.

x-intercepts: $x + \frac{8x}{x^2 - 9} = 0,\ x\left(1 + \frac{8}{x^2 - 9}\right) = 0,$

$\therefore x = 0$ or $1 + \frac{8}{x^2 - 9} = 0,\ x^2 - 1 = 0,\ x = \pm 1$.

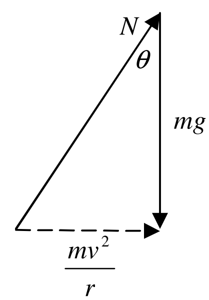


Q3c $x^3 - 4xy + y^3 = 1,\ 3x^2 - 4y - 4x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0,$

$3x^2 - 4y = (4x - 3y^2)\frac{dy}{dx},\ \therefore \frac{dy}{dx} = \frac{3x^2 - 4y}{4x - 3y^2}$.

At $(2,1)$, $m_T = \frac{3(2)^2 - 4(1)}{4(2) - 3(1)^2} = \frac{8}{5}$, $m_N = -\frac{5}{8}$. Equation of the normal is $y - 1 = -\frac{5}{8}(x - 2)$, i.e. $y = -\frac{5}{8}x + \frac{9}{4}$.

Q3d



$N\sin\theta = \frac{mv^2}{r},\quad N\cos\theta = mg,$

$(N\sin\theta)^2 + (N\cos\theta)^2 = \left(\frac{mv^2}{r}\right)^2 + (mg)^2,$

$N^2 = m^2\left(g^2 + \frac{v^4}{r^2}\right),\ \therefore N = m\sqrt{g^2 + \frac{v^4}{r^2}}$.

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Q4ai Volume of a general layer

$= \pi(x+\delta x)^2 y - \pi x^2 y = \pi\left(2x\delta x + (\delta x)^2\right)y = \pi\left(2x\delta x + (\delta x)^2\right)e^{-x^2}$.

Volume of solid $= \lim_{\delta x \to 0} \sum \pi\left(2x\delta x + (\delta x)^2\right)e^{-x^2} = \int_0^N \pi 2xe^{-x^2}\,dx$

Let $u = x^2$, $\frac{du}{dx} = 2x$. Volume $= \int_0^N \pi e^{-u}\frac{du}{dx}\,dx = \int_0^{N^2} \pi e^{-u}\,du$

$= \left[-\pi e^{-u}\right]_0^{N^2} = -\pi e^{-N^2} + \pi = \pi\left(1 - e^{-N^2}\right)$.

Q4aii As $N \to \infty$, volume $\to \pi$.

Q4bi $x^4 + px^3 + qx^2 + rx + s = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\beta\delta)x^2$

$-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta$.

$\therefore \alpha+\beta+\gamma+\delta = -p$, $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r$.

Q4bii Since $\alpha+\beta+\gamma+\delta = -p$, $\therefore (\alpha+\beta+\gamma+\delta)^2 = p^2$,

$\therefore \alpha^2+\beta^2+\gamma^2+\delta^2 + 2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\beta\delta) = p^2$,

$\therefore \alpha^2+\beta^2+\gamma^2+\delta^2 + 2q = p^2$,

$\therefore \alpha^2+\beta^2+\gamma^2+\delta^2 = p^2 - 2q$.

Q4biii For $x^4 - 3x^3 + 5x^2 + 7x - 8 = 0$,

$p^2 - 2q = 9 - 10 = -1 < 0$, i.e. $\therefore \alpha^2+\beta^2+\gamma^2+\delta^2 < 0$. This is impossible because the sum of squares > 0. Hence the quartic function cannot have four real roots.

Q4biv Let $f(x) = x^4 - 3x^3 + 5x^2 + 7x - 8$.

$f(0) = -8$, $f(1) = 2$, $\therefore f(x)$ has at least one real root. According to the fundamental theorem of algebra, $f(x)$ has four real/complex roots. Since $f(x)$ has real coefficients, the roots are pairs of conjugates if they are complex. From part iii and the fact that $f(x)$ has at least one real root, it has exactly two complex roots (conjugates) and two real roots.

Q4ci The normal at $P(x_1, y_1)$ passes through point $B\,(0, -b)$,

$\therefore a^2 y_1(0) - b^2 x_1(-b) = \left(a^2 - b^2\right)x_1 y_1$, $\therefore b^3 x_1 = \left(a^2 - b^2\right)x_1 y_1$.

$\therefore b^3 x_1 - \left(a^2 - b^2\right)x_1 y_1 = 0$, $x_1\left(b^3 - \left(a^2 - b^2\right)y_1\right) = 0$.

Hence $b^3 - \left(a^2 - b^2\right)y_1 = 0$, i.e. $y_1 = \frac{b^3}{a^2 - b^2}$,

or $x_1 = 0$, i.e. $y_1 = \pm b$ because $P(x_1, y_1)$ is on the ellipse.

Q4cii If $y_1 = \frac{b^3}{a^2 - b^2}$ (this includes the possibility that $a = \sqrt{2}b$, i.e. $y_1 = b$), then $0 < y_1 \le b$, $\therefore \frac{b^3}{a^2 - b^2} \le b$,

$\therefore b^3 \le b\left(a^2 - b^2\right)$, $\therefore \frac{b^2}{a^2} \le \frac{1}{2}$. Since $b^2 = a^2\left(1 - e^2\right)$,

$\therefore \frac{b^2}{a^2} = 1 - e^2 \le \frac{1}{2}$, $\therefore e \ge \frac{1}{\sqrt{2}}$.

Q5ai Area of $\Delta ABC = \frac{1}{2}bc = \frac{1}{2}da$, $\therefore bc = da$.

Since $a = \sqrt{b^2 + c^2}$, $\therefore bc = d\sqrt{b^2 + c^2}$ or $b^2c^2 = d^2\left(b^2 + c^2\right)$.

Q5aii $c = \frac{h}{\tan\alpha}$, $b = \frac{h}{\tan\beta}$, $d = \frac{h}{\tan\gamma}$. $b^2c^2 = d^2\left(b^2 + c^2\right)$,

$\frac{h^4}{\tan^2\beta\tan^2\alpha} = \frac{h^2}{\tan^2\gamma}\left(\frac{h^2}{\tan^2\beta} + \frac{h^2}{\tan^2\alpha}\right)$,

$\therefore \frac{1}{\tan^2\beta\tan^2\alpha} = \frac{1}{\tan^2\gamma}\left(\frac{1}{\tan^2\beta} + \frac{1}{\tan^2\alpha}\right)$,

$\tan^2\gamma = \tan^2\beta\tan^2\alpha\left(\frac{1}{\tan^2\beta} + \frac{1}{\tan^2\alpha}\right)$,

$\tan^2\gamma = \tan^2\beta\tan^2\alpha\left(\frac{\tan^2\alpha + \tan^2\beta}{\tan^2\beta\tan^2\alpha}\right)$, hence the result.

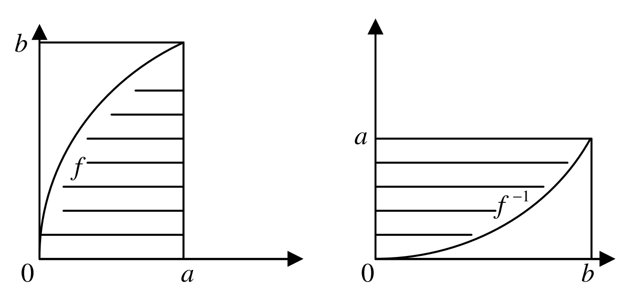
Q5bi Mary is the winner, she is the first to win 5 goals. Ferdinand scores only one goal. There are a total of 6 goals. Mary wins the last goal otherwise Ferdinand would not win any goal at all. Hence Ferdinand scores his only goal in any one of the first five trials.

Q5bii Suppose Mary is the winner, the following table lists a possible outcome of each type and the corresponding number of outcomes.

Type	**Number**
MMMM**M**	1
FMMMM**M**	5
FFMMMM**M**	$\frac{6!}{2!4!} = 15$
FFFMMMM**M**	$\frac{7!}{3!4!} = 35$
FFFFMMMM**M**	$\frac{8!}{4!4!} = 70$

Ferdinand could win in the same manner, $\therefore$ total number of possible outcomes $= 2(1+5+15+35+70) = 252$.

Q5ci The following graphs show $y = f(x)$ and $y = f^{-1}(x)$.



The shaded areas are equal. From the graph of $y = f^{-1}(x)$,

the shaded area + $\int_0^b f^{-1}(x)\,dx = ab$,

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$\therefore \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx = ab,\ \therefore \int_0^a f(x)dx = ab - \int_0^b f^{-1}(x)dx$.

Q5cii $f(x) = \sin^{-1}\left(\frac{x}{4}\right),\ \therefore f(2) = \frac{\pi}{6},\ f^{-1}(x) = 4\sin x$.

$\int_0^2 \sin^{-1}\left(\frac{x}{4}\right)dx = 2\times\frac{\pi}{6} - \int_0^{\frac{\pi}{6}} 4\sin x dx = \frac{\pi}{3} - [-4\cos x]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{3} + 4\cos\frac{\pi}{6} - 4 = \frac{\pi}{3} + 2\sqrt{3} - 4$.

Q5di Base length $AD = 2\sqrt{9-x^2}$,
height $AB = x\tan 60^o = \sqrt{3}x$. Area of ABCD
$= \left(2\sqrt{9-x^2}\right)\left(\sqrt{3}x\right) = 2x\sqrt{3(9-x^2)} = 2x\sqrt{27-3x^2}$.

Q5dii Volume of a general vertical slice of thickness δx
$= 2x\sqrt{27-3x^2}\,\delta x$. Volume of wedge $= \lim_{\delta x\to 0}\sum 2x\sqrt{27-3x^2}\,\delta x$

$= \int_0^3 2x\sqrt{27-3x^2}\,dx = \int_0^3 \sqrt{27-3u}\,\frac{du}{dx}dx = \int_0^9 \sqrt{27-3u}\,du$

$= \left[\frac{2(27-3u)^{\frac{3}{2}}}{3(-3)}\right]_0^9 = 18\sqrt{3}$.

Q6ai For $n = 0$, $I_0(x) = \int_0^x e^{-t}dt = \left[-e^{-t}\right]_0^x = 1 - e^{-x}$,

$I_0(x) = 0!\left(1 - e^{-x}(1)\right) = 1 - e^{-x}$, it is true.
Assume it is true for $n = k$, i.e.

$I_k(x) = \int_0^x t^k e^{-t}dt = k!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^k}{k!}\right)\right]$.

Consider $n = k+1$, $I_{k+1}(x) = \int_0^x t^{k+1}e^{-t}dt$

$= \left[-t^{k+1}e^{-t}\right]_0^x + (k+1)\int_0^x t^k e^{-t}dt = -x^{k+1}e^{-x} + (k+1)I_k(x)$

$= -x^{k+1}e^{-x} + (k+1)k!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^k}{k!}\right)\right]$

$= (k+1)!\left[1 - e^{-x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}\right)\right]$, $\therefore$ it is also true for $n = k+1$. Hence it is true for all $n \ge 0$.

Q6aii Consider $t^n e^{-t}$ in the interval $0 \le t \le 1$.
Since $t^n \ge 0$ and $0 < e^{-t} \le 1$, $\therefore 0 \le t^n e^{-t} \le t^n$. Hence

$0 \le \int_0^1 t^n e^{-t}dt \le \int_0^1 t^n dt,\ 0 \le \int_0^1 t^n e^{-t}dt \le \left[\frac{t^{n+1}}{n+1}\right]_0^1$,

$\therefore 0 \le \int_0^1 t^n e^{-t}dt \le \frac{1}{n+1}$.

Q6aiii From part ii, $0 \le I_n(1) \le \frac{1}{n+1}$,

$\therefore 0 \le n!\left[1 - e^{-1}\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}\right)\right] \le \frac{1}{n+1}$,

$\therefore 0 \le 1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}\right) \le \frac{1}{(n+1)n!}$,

$\therefore 0 \le 1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}\right) \le \frac{1}{(n+1)!}$.

Q6aiv As $n \to \infty$, $\frac{1}{(n+1)!} \to 0$,

$1 - e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}\right) \to 0$,

$e^{-1}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}\right) \to 1$,

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!} \to e$.

Q6bi After expanding,
$LHS = -\left(1 + w + w^2 + w^3 + ... + w^{n-1}\right) + nw^n$.
Since $w^n = 1$, $1 - w^n = 0$,
$\therefore (1-w)\left(1 + w + w^2 + w^3 + ... + w^{n-1}\right) = 0$, and given $w \ne 1$,
$\therefore 1 + w + w^2 + w^3 + ... + w^{n-1} = 0$.
Hence $LHS = nw^n = n$.

Q6bii Let $z = \cos\theta + i\sin\theta$, $z^2 = \cos 2\theta + i\sin 2\theta$,
$z^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$.

Since $\frac{1}{z^2-1} = \frac{z^{-1}}{z - z^{-1}}$,

$\therefore \frac{1}{\cos 2\theta + i\sin 2\theta - 1} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta}$.

Q6biii Let $w = \cos 2\theta + i\sin 2\theta$, where $\theta = \frac{\pi}{n}$.

Then $\frac{1}{w-1} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta} = -\frac{1}{2} + \frac{1}{2}i\cot\theta$.

$\mathrm{Re}\left(\frac{1}{w-1}\right) = -\frac{1}{2}$.

Q6biv From part i, $1 + 2w + 3w^2 + 4w^3 + ... + nw^{n-1} = n\times\frac{1}{w-1}$.

$\mathrm{Re}\left(1 + 2w + 3w^2 + 4w^3 + ... + nw^{n-1}\right) = \mathrm{Re}\left(n\times\frac{1}{w-1}\right)$.

When $n = 5$,

$1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5} = -\frac{5}{2}$.

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Q6bv $1+2\cos\frac{2\pi}{5}+3\cos\frac{4\pi}{5}+4\cos\frac{6\pi}{5}+5\cos\frac{8\pi}{5}$

$=1+2\cos\frac{2\pi}{5}-3\cos\frac{\pi}{5}-4\cos\frac{\pi}{5}+5\cos\frac{2\pi}{5}$

$=1+7\cos\frac{2\pi}{5}-7\cos\frac{\pi}{5}=1+7\left(2\cos^2\frac{\pi}{5}-1\right)-7\cos\frac{\pi}{5}$

$=14\cos^2\frac{\pi}{5}-7\cos\frac{\pi}{5}-6=-\frac{5}{2}$.

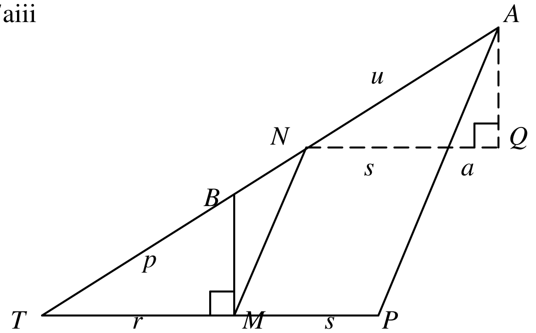
$\therefore 28\cos^2\frac{\pi}{5}-14\cos\frac{\pi}{5}-7=0$ or $4\cos^2\frac{\pi}{5}-2\cos\frac{\pi}{5}-1=0$.

Solve the quadratic equation for $\cos\frac{\pi}{5}$, where $\cos\frac{\pi}{5}>0$, to obtain $\cos\frac{\pi}{5}=\frac{2+\sqrt{20}}{8}=\frac{1+\sqrt{5}}{4}$.

Q7ai Sum of opposite angles = 180°, $\therefore BNPM$ is cyclic.

Q7aii $\angle BNM=\angle BPM$, because $BNPM$ is cyclic and both angles are subtended by the same arc BM.
$\angle BPM=\angle BAP$, because PT is a tangent and the alternate segment theorem follows.
Hence $\angle BNM=\angle BAP$, i.e. the corresponding angles are equal, and this is a sufficient condition for $MN \,//\, PA$.

Q7aiii



Dotted lines are added to the given diagram.

ΔTBM and ΔNAQ are similar, $\therefore \frac{s+a}{u}=\frac{r}{p}$,

$\therefore \frac{s}{u}<\frac{r}{p}$.

Q7aiv Since p is the longest side of ΔTBM, $\therefore \frac{r}{p}<1$,

$\therefore \frac{s}{u}<1$ or $s<u$.

Q7bi $V=\frac{2\pi R}{T}$. The satellite is in free fall, $\therefore |\ddot{x}|=\frac{V^2}{R}=\frac{k}{R^2}$,

$\therefore k=V^2R=\frac{4\pi^2R^3}{T^2}$.

Q7bii $\ddot{x}=-\frac{k}{x^2}$, $\therefore \frac{1}{2}\frac{d}{dx}v^2=-\frac{k}{x^2}$. Moving from R to x,

$\Delta(v^2)=\int_R^x -\frac{2k}{x^2}dx=\left[\frac{2k}{x}\right]_R^x=2k\left(\frac{1}{x}-\frac{1}{R}\right)$

$=\frac{8\pi^2R^3}{T^2}\left(\frac{R-x}{xR}\right)=\frac{8\pi^2R^2}{T^2}\left(\frac{R-x}{x}\right)$.

Since $v=0$ at $x=R$, $\therefore v^2=0+\Delta(v^2)=\frac{8\pi^2R^2}{T^2}\left(\frac{R-x}{x}\right)$.

Q7biii $v=-\frac{2\sqrt{2}\pi R}{T}\sqrt{\frac{R-x}{x}}$, i.e. $\frac{dx}{dt}=-\frac{2\sqrt{2}\pi R}{T}\sqrt{\frac{R-x}{x}}$,

$\frac{dt}{dx}=-\frac{T}{2\sqrt{2}\pi R}\sqrt{\frac{x}{R-x}}$. Note: x is from the centre whilst v is towards the centre, hence the negative sign. Moving from R to 0, the time required $\Delta t=-\frac{T}{2\sqrt{2}\pi R}\int_R^0\sqrt{\frac{x}{R-x}}dx$

$=-\frac{T}{2\sqrt{2}\pi R}\left[R\sin^{-1}\sqrt{\frac{x}{R}}-\sqrt{x(R-x)}\right]_R^0$

$=\frac{T}{2\sqrt{2}\pi R}\times R\sin^{-1}(1)=\frac{T}{4\sqrt{2}}$.

Q8ai Given $a,b,x>0$, $\therefore f(x)>0$.

Define $g(x)=(f(x))^3=\frac{(a+b+x)^3}{27abx}$. Their stationary points occur at the same x value.

$g'(x)=\frac{(a+b+x)^3-3x(a+b+x)^2}{27abx}=0$,

$\therefore (a+b+x)^3-3x(a+b+x)^2=0$,

$(a+b+x)^2[(a+b+x)-3x]=0$, since $a+b+x\neq 0$,

$(a+b+x)-3x=0$ or $x=\frac{a+b}{2}$. The point at $x=\frac{a+b}{2}$ is the only stationary point, it must be the required minimum.

Q8aii $g(x)$ is minimum at $x=\frac{a+b}{2}$, $\therefore g(c)\geq g\left(\frac{a+b}{2}\right)$,

$\therefore g(c)=(f(c))^3=\left(\frac{a+b+c}{3\times\sqrt[3]{abc}}\right)^3\geq\left(\frac{a+b+\frac{a+b}{2}}{3\times\sqrt[3]{ab\frac{a+b}{2}}}\right)^3$

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$$= \left(\frac{\frac{(a+b)}{2}}{\sqrt[3]{\frac{ab(a+b)}{2}}} \right)^3 = \frac{(a+b)^2}{4ab} = \left(\frac{a+b}{2\sqrt{ab}} \right)^2.$$ Since $\frac{a+b}{2} \ge \sqrt{ab}$,

$\therefore \left(\frac{a+b}{2\sqrt{ab}} \right)^2 \ge 1$, $\left(\frac{a+b+c}{3 \times \sqrt[3]{abc}} \right)^3 \ge 1$, $\therefore \frac{a+b+c}{3 \times \sqrt[3]{abc}} \ge 1$.

Hence $\frac{a+b+c}{3} \ge \sqrt[3]{abc}$.

Q8aiii Let a, b and c be the positive real roots.
$x^3 - px^2 + qx - r = (x-a)(x-b)(x-c)$. Expand and compare coefficients, $a+b+c = p$ and $abc = r$.
From part ii, $\frac{a+b+c}{3} \ge \sqrt[3]{abc}$, $\therefore \frac{p}{3} \ge \sqrt[3]{r}$, $\therefore p^3 \ge 27r$.

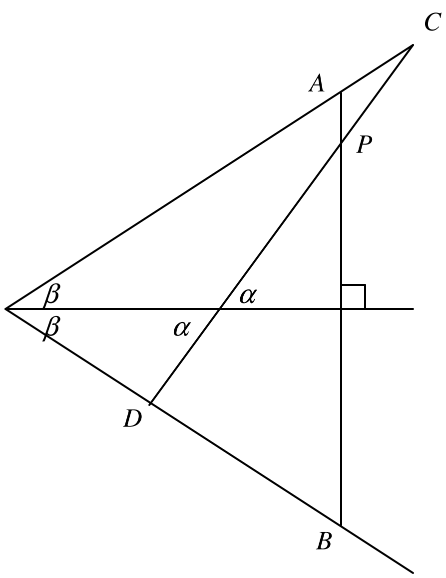
Q8iv For cubic equation $x^3 - 2x^2 + x - 1 = 0$, $p = 2$ and $r = 1$. $\therefore p^3 < 27r$. $\therefore$ the equation does not have 3 positive real roots.

Now consider the impossibility of 1 positive and 2 negative real roots by expressing $x^3 - 2x^2 + x - 1 = 0$ as $x^3 - 2x^2 + x = 1$, $\therefore x(x-1)^2 = 1$. Since $(x-1)^2 \ge 0$ for real x, $\therefore x$ cannot be a negative real number.

According to the fundamental theorem of algebra, and the conjugate roots theorem for real coefficients, the equation has exactly one real root and a pair of conjugate roots.

Q8bi $AP = b\sec\theta - b\tan\theta$, $PB = b\tan\theta + b\sec\theta$,
$\therefore AP \times PB = (b\sec\theta - b\tan\theta)(b\sec\theta + b\tan\theta) = b^2(\sec^2\theta - \tan^2\theta) = b^2$

Q8bii



$\angle ACP = \alpha - \beta$, $\angle PAC = 90 + \beta$, $\angle PBD = 90 - \beta$, $\angle PDB = \alpha + \beta$.
Apply the sine rule to ΔACP and ΔPDB.

$\frac{CP}{\sin\angle PAC} = \frac{AP}{\sin\angle ACP}$, $CP = \frac{AP\sin(90+\beta)}{\sin(\alpha-\beta)} = \frac{AP\cos\beta}{\sin(\alpha-\beta)}$.
$\frac{PD}{\sin\angle PBD} = \frac{PB}{\sin\angle PDB}$, $PD = \frac{PB\sin(90-\beta)}{\sin(\alpha+\beta)} = \frac{PB\cos\beta}{\sin(\alpha+\beta)}$.

Q8biii
$CP \times PD = \frac{AP \times PB\cos^2\beta}{\sin(\alpha-\beta)\sin(\alpha+\beta)} = \frac{b^2\cos^2\beta}{\sin(\alpha-\beta)\sin(\alpha+\beta)}$.
$\therefore CP \times PD$ is dependent on α but independent of P (does not appear in the expression), b and β are constant values that define the hyperbola.

Q8biv The result of part iii suggests that the same is also true for another point, Q, on the line and the hyperbola, i.e.
$CP \times PD = CQ \times QD = \frac{b^2\cos^2\beta}{\sin(\alpha-\beta)\sin(\alpha+\beta)}$.
$\therefore p(r+q) = (p+r)q$, $\therefore pr = rq$, $\therefore p = q$.

Q8bv As CD moves closer to UV, $PQ \to 0$, and P, Q and T become the same point when the two lines coincide.
Since $CP = QD$, $\therefore CT = TD$, i.e. T is the midpoint of CD.

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