2005 NSW BOS Mathematics Exam Solutions

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Q1a 3.7

Q1b $x^3 - 27 = x^3 - 3^3 = (x-3)(x^2+3x+9)$

Q1c $4x + \tan x$

Q1d $\frac{(2x-3)}{2} - \frac{(x-1)}{5} = \frac{5(2x-3)-2(x-1)}{10} = \frac{8x-13}{10}$

Q1e $|x-3| \le 1$, $2 \le x \le 4$

Q1f $x^2 = 8(y-1) = 4\times 2(y-1)$, vertex $(0,1)$, focus $(0,3)$

Q2a $\cos\theta = \frac{1}{\sqrt{2}}$, $\theta = \frac{\pi}{4}, \frac{7\pi}{4}$

Q2bi $\frac{d}{dx}(x\sin x) = x\cos x + \sin x$

Q2bii $\frac{d}{dx}\left(\frac{x^2}{x-1}\right) = \frac{2x(x-1)-x^2}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$

Q2ci Let $u = x^3+1$, $\frac{du}{dx} = 3x^2$, $\int \frac{6x^2}{x^3+1}dx$

$= \int \frac{2}{u}\frac{du}{dx}dx = \int \frac{2}{u}du = 2\log_e u + C = 2\log_e(x^3+1) + C$

Q2cii $\int_0^{\frac{\pi}{6}} \cos 3x dx = \left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{6}} = \frac{1}{3}\sin\frac{\pi}{2} - 0 = \frac{1}{3}$

Q2d $y = \log_e x$, $m_T = \frac{dy}{dx} = \frac{1}{x} = \frac{1}{e}$

Equation of tangent: $y - 1 = \frac{1}{e}(x-e)$, $\therefore y = \frac{1}{e}x$

Q3a $\sum_{n=3}^{5}(2n+1) = (2\times3+1)+(2\times4+1)+(2\times5+1) = 27$

Q3bi $\cos\theta = \frac{7^2+8^2-13^2}{2\times7\times8}$, $\theta = 120^o$

Q3bii $Area = \frac{1}{2}\times7\times8\sin 120^o = 24.2\text{ cm}^2$

Q3ci $m_{AD} = m_{BC} = \frac{6-0}{12-9} = 2$,

Equation of line AD: $y - 0 = 2(x-6)$, $y = 2x - 12$

Q3cii Solve $x - 2y = 0$ and $y = 2x-12$ simultaneously to find the intersection D: $x - 2(2x-12) = 0$, $\therefore 3x = 24$, $x = 8$, $y = 4$, D is $(8,4)$.

Q3ciii DE is 3 units long, E is $(11,4)$.

Q3civ $\angle CDE = \angle DOA$ (Corresponding angles)
$\angle DCE = \angle ODA$ (Corresponding angles)
$\therefore \angle DEC = \angle OAD$, $\therefore \Delta OAD$ and ΔDEC are similar.

Q3cv $AD : EC = OA : DE = 6 : 3 = 2 : 1$
or $\frac{AD}{EC} = 2$

Q4ai Length of arc $AB = r\theta = 90\times0.6 = 54$ cm

Q4aii Length of $AB = 2\times90\sin 0.3 = 53.2$ cm

Q4aiii Sector area $= \frac{1}{2}r^2\theta = \frac{1}{2}\times90^2\times0.6 = 2430\text{ cm}^2$

Q4bi $f(x) = (x+3)(x^2-9) = (x+3)(x+3)(x-3) = 0$
$\therefore x = -3$ or 3

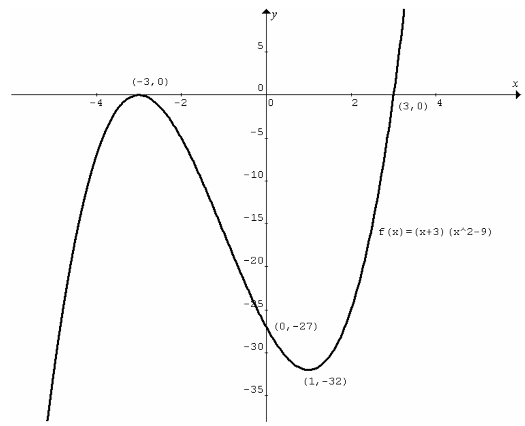
Q4bii $f'(x) = (x^2-9) + 2x(x+3) = (x+3)(3x-3)$

Stationary points: $(x+3)(3x-3) = 0$, $\therefore x = -3$ and $y = 0$ or $x = 1$ and $y = -32$

Nature of stationary points:

At $x =$	-4	-3	0	1	2
$f'(x)$	>0	$=0$	<0	$=0$	>0
Nature		Local max.		Local min.	

Q4biii



Q4biv $f'(x) = 3x^2 + 6x - 9$, $f''(x) = 6x + 6 = 0$, $x = -1$.
$f''(x) < 0$ for $x < -1$, $\therefore f(x)$ is concave down for $x < -1$.

Q5a $\log_3 7 = \frac{\log_{10}7}{\log_{10}3} = 1.77$

Q5b $\angle ADC = 180 - 120 = 60^o$ ($\angle ADC, \angle DAB$ are co-interior angles). $\therefore \angle BCE = \angle ADC = 60^o$ (Corresponding angles).
$BC = AD = BE$, $\therefore \Delta BCE$ is isosceles and
$\angle BEC = \angle BCE = 60^o$. $\therefore \angle CBE = 60^o$. Hence ΔBCE is equilateral.

Q5c At point P, $\frac{dy}{dx} = 2e^x + 3 = 5$, $\therefore e^x = 1$, $x = 0$ and $y = 2$. P is $(0,2)$.

Q5di $\Pr(all.red) = \frac{^{100}C_3}{^{300}C_3} = 0.0363$

Q5dii $\Pr(at.least.one.not.red) = 1 - \Pr(all.red) = 1 - 0.0363 = 0.9637$

Q5diii $\Pr(one.of.each.colour) = \frac{(^{100}C_1)^3}{^{300}C_3} = 0.2245$

Q6a
$\int_0^{20} f(x)dx \approx \frac{20-0}{3\times 4}\left[f(0)+4f(5)+2f(10)+4f(15)+f(20)\right]$
$=\frac{5}{3}\left[15+100+44+72+10\right]=\frac{1205}{3}\approx 402$

Q6bi At $t=10$ min, $V=3600\left(1-\frac{1}{6}\right)^2=2500$ litres

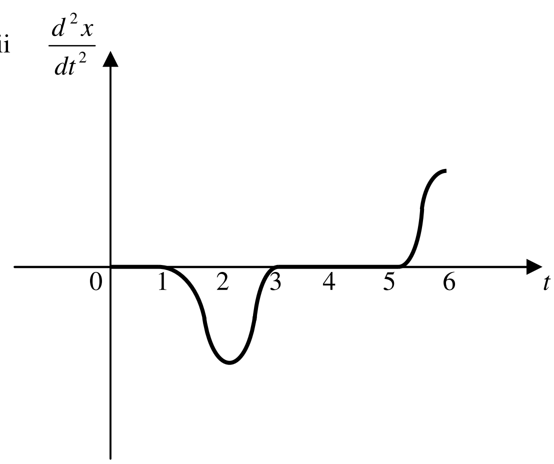
Q6bii At $t=20$ min, $\frac{dV}{dt}=2\times\frac{-1}{60}\times 3600\left(1-\frac{t}{60}\right)$
$=-120\left(1-\frac{t}{60}\right)=-120\left(1-\frac{20}{60}\right)=-80$
Drainage rate $=80$ litres per min.

Q6biii $\frac{dV}{dt}=2t-120$, the fastest rate is -120 that occurs at $t=0$.

Q6ci Solve $y=x^2$ and $y=12-2x^2$ simultaneously:
$x^2=12-2x^2$, $3x^2=12$, $x=-2$ and $y=4$, or $x=2$ and $y=4$. The intersecting points are $(-2,4)$ and $(2,4)$.

Q6cii $Volume=\int_0^4 \pi(x_1)^2\,dy+\int_4^{12}\pi(x_2)^2\,dy$
$=\int_0^4 \pi y\,dy+\int \pi\frac{1}{2}(12-y)dy=\left[\frac{\pi y^2}{2}\right]_0^4+\left[\frac{\pi(12-y)^2}{-4}\right]_4^{12}$
$=8\pi+16\pi=24\pi$ cubic units

Q7ai $a=50000$, $d=2500$, $t_{13}=50000+12(2500)=80000$
Q7aii $a=50000$, $r=1.04$, $t_{13}=50000(1.04)^{12}=80051.61$
Q7aiii Kay: $S_{20}=\frac{50000(1.04^{20}-1)}{0.04}=1488903.93$
Anne: $S_{20}=\frac{20}{2}(2(50000)+19(2500))=1475000$
Kay's total exceeds Anne's total by
$1488903.93-1475000=\$13903.93$
Q7bi $t=2$
Q7bii $t=4$
Q7biii



Q8ai $x^2=R^2-h^2$, $V=\pi x^2(2h)=2\pi h(R^2-h^2)$
Q8aii Max. volume when
$\frac{dV}{dh}=(R^2-h^2)(2\pi)+(-2h)(2\pi h)=0$,
$2\pi(R^2-3h^2)=0$, $\therefore h=\frac{R}{\sqrt{3}}$.

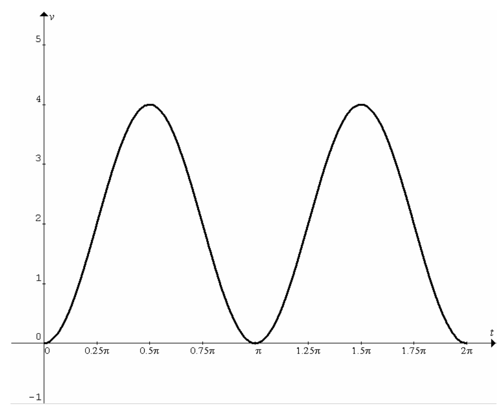
Q8b Area of shaded region = area of quarter circle −
$\int_0^1(x^2-3x+2)dx=\pi-\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_0^1=\pi-\frac{5}{6}$

Q8ci $A_1=(3\times10^6)(1.12)-4.8\times10^5$,
$A_2=A_1(1.12)-4.8\times10^5=((3\times10^6)(1.12)-4.8\times10^5)(1.12)-4.8\times10^5$
$=(3\times10^6)(1.12)^2-(4.8\times10^5)(1+1.12)$

Q8cii $A_n=(3\times10^6)(1.12)^n-(0.48\times10^6)(1+1.12+...+1.12^{n-1})$
$=10^6\left(3(1.12)^n-0.48\left(\frac{1(1.12^n-1)}{0.12}\right)\right)$
$=10^6(3(1.12)^n-4(1.12)^n+4)=10^6\left[4-(1.12)^n\right]$

Q8ciii After the final payment, $A_n=0$, $10^6\left[4-(1.12)^n\right]=0$,
$\therefore (1.12)^n=4$, $n=\frac{\log_e 4}{\log_e 1.12}=12.23$, $\therefore$ year 2017.

Q9ai At $t=0$, $x=0$, $\dot{x}=0$.
At time t, $\dot{x}=\int 4\sin 2t\,dt=-2\cos 2t+C$, $\therefore 0=-2+C$,
i.e. $C=2$. $\therefore \dot{x}=2-2\cos 2t$
Q9aii



First comes to rest at $t=\pi$.

Q9aiii Distance = magnitude of displacement = area under v-t graph from $t=0$ to $t=\pi$.
$\int_0^{\pi}(2-2\cos 2t)dt=\left[2t-\sin 2t\right]_0^{\pi}=2\pi$

Q9bi $BD=6\sin\theta$, $\angle DBE=\angle EDF=\angle FEG$ etc.
$DE=BD\sin\theta=6\sin^2\theta$, $EF=DE\sin\theta=6\sin^3\theta$.

Q9bii $BD+EF+GH+...=6\sin\theta+6\sin^3\theta+6\sin^5\theta+...$
$=6(\sin\theta+\sin^3\theta+\sin^5\theta+...)=6\left(\frac{\sin\theta}{1-\sin^2\theta}\right)=6\left(\frac{\sin\theta}{\cos^2\theta}\right)$
$=6\left(\frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}\right)=6\sec\theta\tan\theta$

Q10ai $m = \dfrac{\alpha^2 - \beta^2}{\alpha - \beta} = \dfrac{(\alpha-\beta)(\alpha+\beta)}{(\alpha-\beta)} = \alpha + \beta$

$A(\alpha, \alpha^2)$ lies on the line $y = mx + b$, $\therefore \alpha^2 = (\alpha+\beta)\alpha + b$,

$\therefore \alpha\beta = -b$

Q10aii

$AB = \sqrt{(\alpha-\beta)^2 + (\alpha^2 - \beta^2)^2} = \sqrt{(\alpha-\beta)^2\left[1 + (\alpha+\beta)^2\right]}$

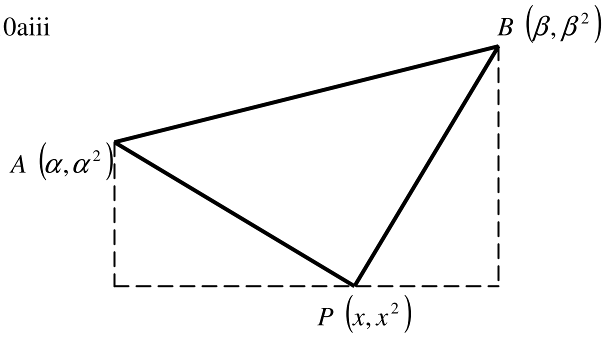
Since $\alpha + \beta = m$, $\therefore m^2 = (\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\therefore m^2 = \beta^2 - 2\alpha\beta + \alpha^2 + 4\alpha\beta = (\beta-\alpha)^2 - 4b$

$\therefore (\alpha-\beta)^2 = (\beta-\alpha)^2 = m^2 + 4b$

Hence $AB = \sqrt{(m^2+4b)(1+m^2)}$

Q10aiii



Area of ΔABP = Area of trapezium − areas of two smaller Δs

$= \frac{1}{2}\left[(\alpha^2 - x^2) + (\beta^2 - x^2)\right](\beta - \alpha) - \frac{1}{2}(\alpha^2 - x^2)(x - \alpha) - \frac{1}{2}(\beta^2 - x^2)(\beta - x)$

$= \dfrac{1}{2}\left((\alpha^2 - x^2)(\beta - x) + (\beta^2 - x^2)(x - \alpha)\right)$ expand and simplify

$= \dfrac{1}{2}\left((\beta^2 - \alpha^2)x - (\beta-\alpha)x^2 - (\beta-\alpha)\alpha\beta\right)$

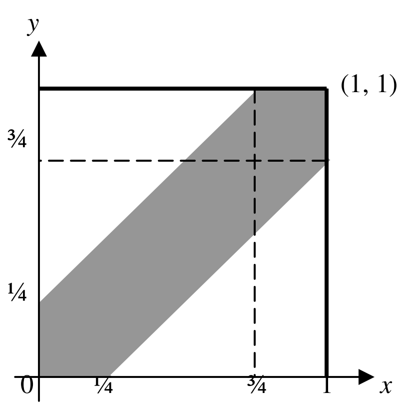
$= \dfrac{1}{2}\left((\beta+\alpha)x - x^2 - \alpha\beta\right)(\beta - \alpha)$

$= \dfrac{1}{2}(mx - x^2 + b)\sqrt{m^2 + 4b}$

Q10aiv $\dfrac{dA}{dx} = \dfrac{1}{2}\sqrt{m^2+4b}\,(m - 2x) = 0$, $\therefore x = \dfrac{m}{2}$ and

$y = \dfrac{m^2}{4}$. Hence P is $\left(\dfrac{m}{2}, \dfrac{m^2}{4}\right)$.

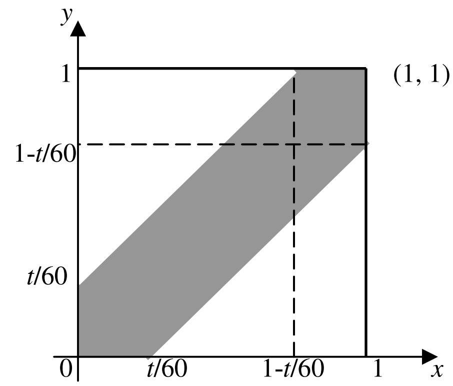
Q10bi



The shaded region satisfies all the inequalities and is inside the square. $\therefore$ they will meet.

Q10bii $\Pr(meet) = area\ of\ shaded\ region = 1 - \left(\dfrac{3}{4}\right)^2 = \dfrac{7}{16}$

Q10biii



$\Pr(meet) = 1 - \left(1 - \dfrac{t}{60}\right)^2 = 0.5$, $\left(1 - \dfrac{t}{60}\right)^2 = 0.5$,

$1 - \dfrac{t}{60} = 0.7071$, $t = 17.6$ minutes.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.

