2008 NSW BOS Mathematics Exam Solutions

Q1a 1.62

Q1b $3x^2+x-2=(3x-2)(x+1)$

Q1c $\frac{2}{n}-\frac{1}{n+1}=\frac{2(n+1)-n}{n(n+1)}=\frac{n+2}{n(n+1)}$

Q1d $|4x-3|=7 \Rightarrow 4x-3=7$ or $-(4x-3)=7$,
i.e. $x=\frac{5}{2}$ or -1

Q1e $(\sqrt{3}-1)(2\sqrt{3}+5)=6+5\sqrt{3}-2\sqrt{3}-5=3\sqrt{3}+1$

Q1f $S_n=\frac{n}{2}(2a+(n-1)d)$, where $a=3$, $d=4$ and $n=21$.
$S_{21}=\frac{21}{2}(2(3)+20(4))=903$

Q2ai $\frac{d}{dx}(x^2+3)^9=9(x^2+3)^8(2x)=18x(x^2+3)^8$

Q2aii $\frac{d}{dx}(x^2\log_e x)=2x(\log_e x)+x^2\left(\frac{1}{x}\right)=x(2\log_e x+1)$

Q2aiii $\frac{d}{dx}\left(\frac{\sin x}{x+4}\right)=\frac{(x+4)\cos x-\sin x}{(x+4)^2}$

Q2bi $M\left(\frac{-1+5}{2},\frac{4+8}{2}\right)$, i.e. $M(2,6)$.

Equation of the line through M with gradient $-\frac{1}{2}$:
$y-6=-\frac{1}{2}(x-2)$, $2y-12=-x+2$, $\therefore x+2y-14=0$.

Q2ci $\int\frac{dx}{x+5}=\log_e|x+5|+c$

Q2cii $\int_0^{\frac{\pi}{12}}\sec^2 3x\,dx=\left[\frac{1}{3}\tan 3x\right]_0^{\frac{\pi}{12}}=\frac{1}{3}\tan\frac{\pi}{4}=\frac{1}{3}$

Q3ai Gradient of line $BC=\frac{5-3}{1-0}=2$, gradient of line AD is also 2. $\therefore$ line BC and line AD are parallel. Hence $ABCD$ is a trapezium.

Q3aii The y-coordinate of D is 5. Substitute $y=5$ in $2x-y-1=0$ to obtain $x=3$. Hence $D(3,5)$.

Q3aiii Length of $BC=\sqrt{(1-0)^2+(5-3)^2}=\sqrt{5}$

Q3aiv Gradient of perpendicular $=-\frac{1}{gradAD}=-\frac{1}{2}$.

Equation of perpendicular from B to AD: $y-3=-\frac{1}{2}x$.

Solve $y-3=-\frac{1}{2}x$ and $2x-y-1=0$ simultaneously to find the intersection. $(2x-1)-3=-\frac{1}{2}x$, $\therefore x=\frac{8}{5}$ and $y=\frac{11}{5}$.

Perpendicular distance from B to $AD=\sqrt{\left(\frac{8}{5}-0\right)^2+\left(\frac{11}{5}-3\right)^2}=\frac{4}{\sqrt{5}}$.

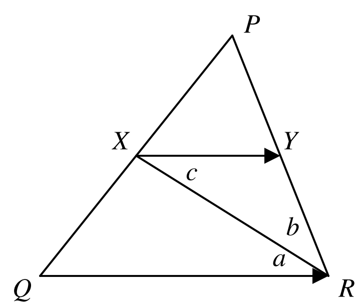
Q3av Length of $AD=\sqrt{(3-0)^2+(5-{}^{-}1)^2}=3\sqrt{5}$,
length of $BC=\sqrt{5}$, perpendicular distance from B to $AD=\frac{4}{\sqrt{5}}$.

Area of $ABCD=\frac{1}{2}(\sqrt{5}+3\sqrt{5})\times\frac{4}{\sqrt{5}}=8$ square units.

Q3bi $\frac{d}{dx}\log_e(\cos x)=\frac{1}{\cos x}\times(-\sin x)=-\tan x$

Q3bii $\int_0^{\frac{\pi}{4}}\tan x\,dx=[-\log_e(\cos x)]_0^{\frac{\pi}{4}}=-\log_e\left(\cos\frac{\pi}{4}\right)+\log_e(\cos 0)$
$=-\log_e\left(\frac{1}{\sqrt{2}}\right)+0=\log_e(\sqrt{2})=\frac{1}{2}\log_e 2$

Q4a



$a=b$ because XR bisects $\angle PRQ$, $c=a$ because $\angle YXR$ and $\angle XRQ$ are alternate angles. $\therefore c=b$ and ΔXYR is an isosceles triangle.

Q4bi $50\times1.2^8=215$ mm

Q4bii $50\times1.2^n>400$, $1.2^n>8$, $n>\frac{\log 8}{\log 1.2}=11.4$.
$\therefore$ 12 times at least.

Q4ci $x^2=8(y-3)$, $x^2=4(2)(y-3)$. Vertex is $(0,3)$.

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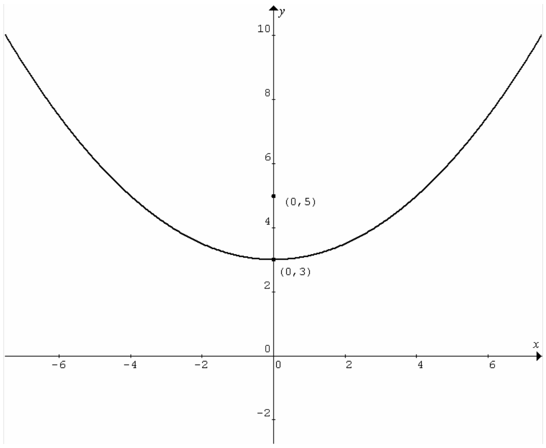
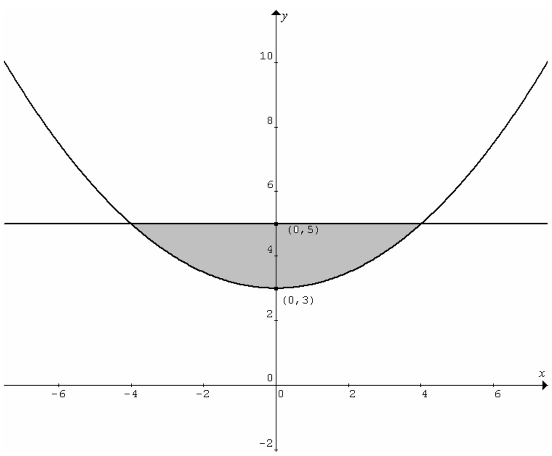
Q4cii Focus is $(0, 3+2)$, i.e. $(0, 5)$.

Q4ciii



Q4civ



x-coordinates of intersections: $x^2 = 8(5-3)$, $x = \pm 4$.

Parabola: $y = \frac{1}{8}x^2 + 3$.

Shaded area $= 2\times\int_0^4 \left(5 - \left(\frac{1}{8}x^2 + 3\right)\right)dx = 2\times\int_0^4 \left(2 - \frac{1}{8}x^2\right)dx$

$= 2\times\left[2x - \frac{x^3}{24}\right]_0^4 = \frac{32}{3}$ square units.

Q5a $\frac{dy}{dx} = 1 - 6\sin 3x$, $y = \int (1 - 6\sin 3x)dx = x + 2\cos 3x + c$.
The curve passes through $(0,7)$, $\therefore 7 = 2\cos 0 + c$, $c = 5$.
Hence $y = x + 2\cos 3x + 5$.

Q5bi $5 + 10x + 20x^2 + 40x^3 + ... = 5\left(1 + (2x) + (2x)^2 + (2x)^3 + ...\right)$.

This series has a limiting sum if $-1 < 2x < 1$. $\therefore -\frac{1}{2} < x < \frac{1}{2}$,

i.e. $|x| < \frac{1}{2}$.

Q5bii $a = 1$, $r = 2x$, $S_\infty = \frac{a}{1-r}$.

$\therefore \frac{1}{1-2x} = \frac{100}{5}$, $1 = 20 - 40x$, $x = \frac{19}{40}$.

Q5c $I = Ae^{-ks}$ and $I = 6000$ when $s = 0$. $\therefore A = 6000$.

Q5cii $I = 6000e^{-ks}$ and $I = 1000$ when $s = 6$.

$\therefore 1000 = 6000e^{-6k}$, $e^{-6k} = \frac{1}{6}$, $e^{6k} = 6$, $6k = \log_e 6$,

$\therefore k = \frac{1}{6}\log_e 6 \approx 0.2986$.

Q5ciii $I = Ae^{-ks}$, $\frac{dI}{ds} = -kAe^{-ks} = -kI$.
When $s = 6$, $I = 1000$,

$\frac{dI}{ds} = -kI = -\left(\frac{1}{6}\log_e 6\right)1000 = -\frac{500}{3}\log_e 6 \approx -298.6$

The light intensity is decreasing at 298.6 lux per metre at 6 metres below the surface of the lake.

Q6a $2\sin^2\frac{x}{3} = 1$ and $-\pi \le x \le \pi$

$1 - 2\sin^2\frac{x}{3} = 0$, $\therefore \cos\frac{2x}{3} = 0$ and $-\frac{2\pi}{3} \le \frac{2x}{3} \le \frac{2\pi}{3}$.

$\therefore \frac{2x}{3} = \pm\frac{\pi}{2}$, $x = \pm\frac{3\pi}{4}$.

Q6bi Read from graph, $v = 20$ ms^{-1}.

Q6bii Read from graph, $t = 10$ s.

Q6biii Read from graph, $t = 6$ s.

Q6biv $A = \frac{1}{3}(20 + 4(37) + 50) + \frac{1}{3}(50 + 4(61) + 70)$

$+ \frac{1}{3}(70 + 4(77) + 80) + \frac{1}{3}(80 + 4(75) + 60) \approx 493$ m

Q6ci $V = \int_3^6 \pi y^2 dx = \pi\int_3^6 \frac{25}{(x-2)^2}dx = 25\pi\left[-\frac{1}{x-2}\right]_3^6$

$= 25\pi\left(-\frac{1}{4} + 1\right) = \frac{75\pi}{4}$ cubic units.

Q7a $\log_e x - \frac{3}{\log_e x} = 2$, $(\log_e x)^2 - 2\log_e x - 3 = 0$,

$(\log_e x - 3)(\log_e x + 1) = 0$.

$\therefore \log_e x = 3$, $x = e^3$, or $\log_e x = -1$, $x = \frac{1}{e}$.

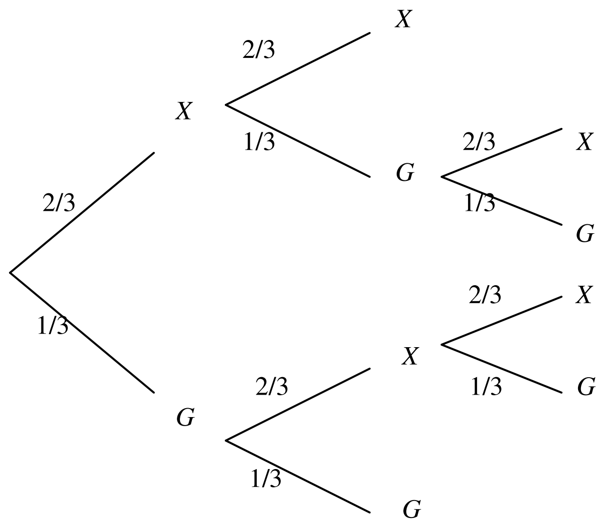
Q7bi $r\theta=\frac{10\pi}{3}$, $\therefore \theta=\frac{10\pi}{3r}$.

Since $0<\theta\le 2\pi$, $\therefore 0<\frac{10\pi}{3r}\le 2\pi$, $0<\frac{5}{3r}\le 1$,

$0<5\le 3r$, $\therefore r\ge\frac{5}{3}$.

Q7bii Area $=\frac{1}{2}r^2\theta=\frac{1}{2}r(r\theta)=\frac{1}{2}(4)\left(\frac{10\pi}{3}\right)=\frac{20\pi}{3}$ square units

Q7ci



Possible outcomes are *XX*, *XGX*, *XGG*, *GXX*, *GXG*, *GG*.

Q7cii $\Pr(XGG)+\Pr(GXG)+\Pr(GG)$

$=\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}=\frac{7}{27}$.

Q7ciii $\Pr(3games)=1-\Pr(2games)=1-\left(\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2\right)=\frac{4}{9}$.

Q8ai $f(x)=0$, $x^4-8x^2=0$, $x^2(x^2-8)=0$. $\therefore x=0$ or $\pm 2\sqrt{2}$. The graph crosses the *y*-axis at $(0,0)$, or crosses the *x*-axis at $(-2\sqrt{2},0)$ or $(2\sqrt{2},0)$.

Q8aii $f(x)=x^4-8x^2$,

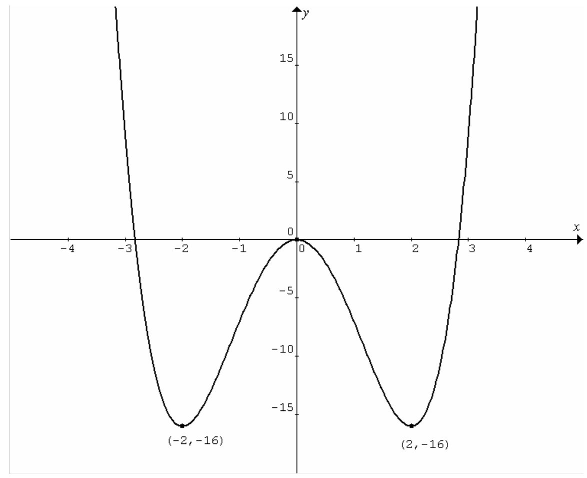
$f(-x)=(-x)^4-8(-x)^2=x^4-8x^2=f(x)$,

$\therefore f(x)$ is an even function.

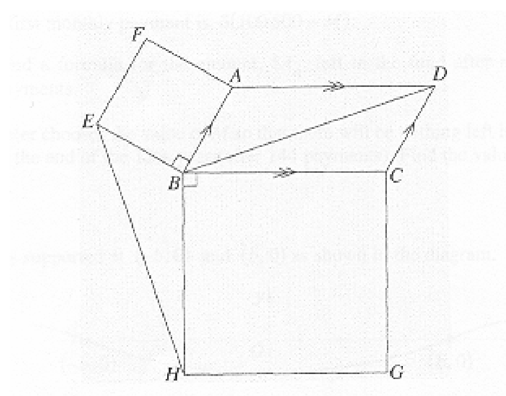
Q8aiii $f'(x)=4x^3-16x=0$, $4x(x^2-4)=0$, $\therefore x=0$ or ± 2.

x		-2		0		2	
y		-16		0		-16	
(x,y)		$(-2,-16)$		$(0,0)$		$(2,-16)$	
dy/dx	$-$	0	$+$	0	$+$	0	$-$
Nature		Local min		Local max		Local min	

Q8aiv



Q8bi



$CD=BA$ (parallelogram), $BA=BE$ (square), $\therefore CD=BE$.

Q8bii $\angle HBE+\angle ABC=180°=\angle ABC+\angle BCD$,

$\therefore \angle HBE=\angle BCD$, $HB=BC$ (square) and $CD=BE$ (Q8bi).

Hence $\Delta HBE\equiv\Delta BCD$. $\therefore BD=HE$.

Q9ai $\Pr(neither)=0.15\times 0.15=0.0225$

Q9aii $\Pr(own\cap use)=\Pr(use\mid own)\Pr(own)$

$=0.20\times 0.85=0.17$

Q9bi $P=100000$, $R=1+\frac{6}{100\times 12}=1.005$,

monthly payment $=M$.

$A_n=100000(1.005^n)-\frac{M(1.005^n-1)}{0.005}$

Q9bii $100000(1.005^{144})-\frac{M(1.005^{144}-1)}{0.005}=0$,

$\therefore M=\frac{100000(1.005^{144})\times 0.005}{1.005^{144}-1}=975.85$

Q9ci $f''(x)=k\left(b^2-x^2\right)$,

$f'(x)=\int k\left(b^2-x^2\right)dx = k\left(b^2x-\frac{x^3}{3}\right)+c$.

Given $f'(-b)=-f'(b)$, $\therefore k\left(-b^3+\frac{b^3}{3}\right)+c=-k\left(b^3-\frac{b^3}{3}\right)-c$,

$\therefore 2c=0$, $c=0$. Hence $f'(x)=k\left(b^2x-\frac{x^3}{3}\right)$.

Q9cii $f(x)=\int k\left(b^2x-\frac{x^3}{3}\right)dx = k\left(\frac{b^2x^2}{2}-\frac{x^4}{12}\right)+C$.

Given $f(b)=0$, $\therefore k\left(\frac{b^4}{2}-\frac{b^4}{12}\right)+C=0$, $\frac{5kb^4}{12}+C=0$,

$\therefore C=-\frac{5kb^4}{12}$ and $f(x)=k\left(\frac{b^2x^2}{2}-\frac{x^4}{12}\right)-\frac{5kb^4}{12}$.

Hence at $x=0$, $f(0)=-\frac{5kb^4}{12}$. The beam is $\frac{5kb^4}{12}$ units below the *x*-axis.

Q10a $y=\log_e(x-2)$, $x=e^y+2$.
Area of shaded region

$=7\log_e 5-\int_0^{\log_e 5} x\,dy = 7\log_e 5-\int_0^{\log_e 5}\left(e^y+2\right)dy$

$=7\log_e 5-\left[e^y+2y\right]_0^{\log_e 5} = 7\log_e 5-\left(e^{\log_e 5}+2\log_e 5\right)+1$

$=5\log_e 5-4$ square units.

Q10bi ΔOKJ and ΔOMP are similar.

$\therefore \frac{PM}{KJ}=\frac{OM}{OJ}$, $\frac{PM}{s}=\frac{\ell-x}{x}$, $PM=\frac{s(\ell-x)}{x}$.

$A=\frac{1}{2}sx\sin\alpha+\frac{1}{2}(\ell-x)\frac{s(\ell-x)}{x}\sin\alpha$

$=s\left(\frac{x}{2}+\frac{\ell^2-2\ell x+x^2}{2x}\right)\sin\alpha$

$=s\left(x-\ell+\frac{\ell^2}{2x}\right)\sin\alpha$, where $0<x<\ell$.

Q10bii $\frac{dA}{dx}=s\left(1-\frac{\ell^2}{2x^2}\right)\sin\alpha=0$, $1-\frac{\ell^2}{2x^2}=0$,

$x^2=\frac{\ell^2}{2}$, $x=\frac{\ell}{\sqrt{2}}$.

When $x<\frac{\ell}{\sqrt{2}}$, $x^2<\frac{\ell^2}{2}$, $1<\frac{\ell^2}{2x^2}$, $1-\frac{\ell^2}{2x^2}<0$.

$\therefore \frac{dA}{dx}<0$, since s and $\sin\alpha$ are both positive.

When $x>\frac{\ell}{\sqrt{2}}$, $x^2>\frac{\ell^2}{2}$, $1>\frac{\ell^2}{2x^2}$, $1-\frac{\ell^2}{2x^2}>0$.

$\therefore \frac{dA}{dx}>0$, since s and $\sin\alpha$ are both positive.

$\therefore$ A is a minimum at $x=\frac{\ell}{\sqrt{2}}$.

Q10biii From Q10bi,

$PM=\frac{s(\ell-x)}{x}=s\left(\frac{\ell}{x}-1\right)=s\left(\sqrt{2}-1\right)$ metres.

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