**NSW BOS Mathematics Extension 2 Solutions 2008**
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Q1a Let $u = 5 + x^3$, $\frac{du}{dx} = 3x^2$, $\int \frac{x^2}{(5+x^3)^2} dx = \int \frac{1}{3u^2} \frac{du}{dx} du$

$= \int \frac{1}{3u^2} du = -\frac{1}{3u} + c = -\frac{1}{3(5+x^3)} + c$.

Q1b $\int \frac{dx}{\sqrt{4x^2+1}} = \int \frac{dx}{\sqrt{(2x)^2+1}} = \frac{1}{2}\ln\left(2x + \sqrt{4x^2+1}\right) + c$.

Q1c

$\int_0^1 \tan^{-1} x dx = \left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx = \left[x\tan^{-1} x\right]_0^1 - \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1$

$= \left[x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$.

Q1d Let $u = \sqrt{2x-1}$ for $2x-1>0$, $\frac{du}{dx} = \frac{1}{\sqrt{2x-1}}$,

$x = \frac{1}{2}\left(u^2+1\right)$.

When $x=1$, $u=1$; when $x=2$, $u=\sqrt{3}$.

$\int_1^2 \frac{dx}{x\sqrt{2x-1}} = \int_1^2 \frac{1}{\frac{1}{2}\left(u^2+1\right)} \frac{du}{dx} dx = \int_1^{\sqrt{3}} \frac{2}{1+u^2} du$

$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$.

Q1e $\int_0^1 \frac{8(1-x)}{\left(2-x^2\right)\left(2-2x+x^2\right)} dx = \int_0^1 \left(\frac{4-2x}{2-2x+x^2} + \frac{-2x}{2-x^2}\right) dx$

$= \int_0^1 \left(\frac{-(-2+2x)+2}{2-2x+x^2} + \frac{-2x}{2-x^2}\right) dx$

$= \int_0^1 \left(\frac{-(-2+2x)}{2-2x+x^2} + \frac{2}{2-2x+x^2} + \frac{-2x}{2-x^2}\right) dx$

$= \int_0^1 \left(\frac{-(-2+2x)}{2-2x+x^2} + \frac{2}{1+(1-x)^2} + \frac{-2x}{2-x^2}\right) dx$

$= \left[-\ln\left(2-2x+x^2\right) - 2\tan^{-1}(1-x) + \ln\left(2-x^2\right)\right]_0^1$

$= \ln 2 + 2\tan^{-1} 1 - \ln 2 = \frac{\pi}{2}$.

Q2a $(1+2i)(1-3i) = a+ib$, $1-3i+2i-6i^2 = a+ib$,

$7-i = a+ib$, $\therefore a=7$, $b=-1$.

Q2bi

$\frac{1+i\sqrt{3}}{1+i} = \frac{1+i\sqrt{3}}{1+i} \times \frac{1-i}{1-i} = \frac{1-i+i\sqrt{3}+\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2} + i\frac{\sqrt{3}-1}{2}$.

Q2bii $1+i\sqrt{3} = 2cis\frac{\pi}{3}$, $1+i = \sqrt{2}cis\frac{\pi}{4}$.

$\frac{1+i\sqrt{3}}{1+i} = \frac{2cis\frac{\pi}{3}}{\sqrt{2}cis\frac{\pi}{4}} = \sqrt{2}cis\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sqrt{2}cis\frac{\pi}{12}$.

Q2biii $\frac{1+i\sqrt{3}}{1+i} = \sqrt{2}cis\frac{\pi}{12}$,

$\therefore \frac{1+\sqrt{3}}{2} + i\frac{\sqrt{3}-1}{2} = \sqrt{2}\cos\frac{\pi}{12} + i\sqrt{2}\sin\frac{\pi}{12}$,

$\therefore \sqrt{2}\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2}$. Hence $\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

Q2biv $\left(\frac{1+i\sqrt{3}}{1+i}\right)^{12} = \left(\sqrt{2}cis\frac{\pi}{12}\right)^{12} = 2^6 cis\left(12 \times \frac{\pi}{12}\right) = 2^6 cis\pi$

$= 2^6\cos\pi = -64$.

Q2c $z^2 + \bar{z}^2 = 8$, $x^2 + i2xy - y^2 + x^2 - i2xy - y^2 = 8$,

$\therefore \frac{x^2}{4} - \frac{y^2}{4} = 1$. Rectangular hyperbola centred at the origin with asymptotes $y = \pm x$ and intercepts $(-2,0)$ and $(2,0)$.

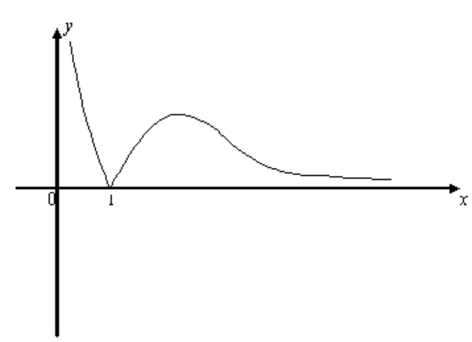
Q2di $M$ is $\frac{1}{2}\left(\omega z + \bar{\omega} z\right)$

$= \frac{z}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right) = z\cos\frac{2\pi}{3} = -\frac{z}{2}$.

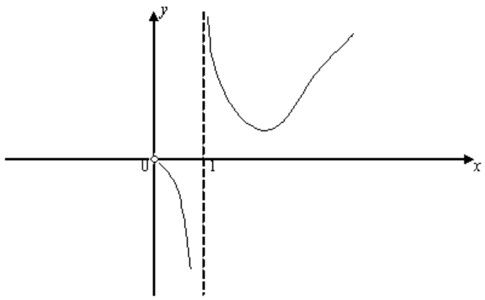
Q2dii Diagonals of a parallelogram bisect each other. *M* is the midpoint of diagonal *PS*.
Let $\alpha$ be the complex number represented by *S*.

$\therefore \frac{1}{2}(z+\alpha) = -\frac{z}{2}$. Hence $\alpha = -2z$.

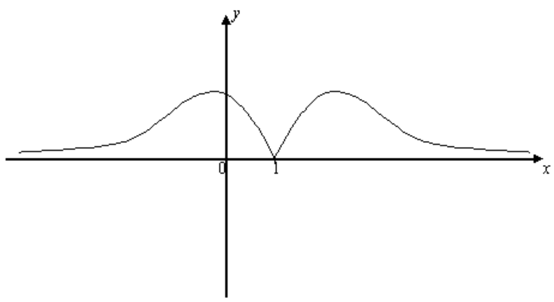
Q3ai $y = |g(x)|$



Q3aii $y = \dfrac{1}{g(x)}$




Q3aiii $y = f(x) = \begin{cases} g(x), & x \ge 1 \\ g(2-x), & x < 1 \end{cases}$




Q3bi Since $z^2 \ge 0$ and $z^4 \ge 0$ for real $z$,
$\therefore p(z) = 1 + z^2 + z^4 > 0$ for real $z$. Hence $p(z)$ has no real zeros.

Q3bii $\alpha$ is a zero of $p(z)$, $\therefore 1 + \alpha^2 + \alpha^4 = 0$,
$\therefore (1-\alpha^2)(1+\alpha^2+\alpha^4) = 0$, $1 - \alpha^6 = 0$. Hence $\alpha^6 = 1$.

Q3biii $p(\alpha^2) = 1 + (\alpha^2)^2 + (\alpha^2)^4 = 1 + \alpha^4 + \alpha^8 = 1 + \alpha^4 + \alpha^6\alpha^2$
$= 1 + \alpha^2 + \alpha^4 = 0$, $\therefore \alpha^2$ is a zero of $p(z)$.

Q3ci For $n \ge 0$, $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^{2n}\theta d\theta$.

For $n \ge 1$, $I_n + I_{n-1} = \displaystyle\int_0^{\frac{\pi}{4}} \left(\tan^{2n}\theta + \tan^{2n-2}\theta\right) d\theta$

$= \displaystyle\int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta\left(\tan^2\theta + 1\right) d\theta = \int_0^{\frac{\pi}{4}} \tan^{2n-2}\theta \sec^2\theta d\theta$.

Let $u = \tan\theta$, $\dfrac{du}{d\theta} = \sec^2\theta$, $u = 0$ when $\theta = 0$ and $u = 1$ when

$\theta = \dfrac{\pi}{4}$. $\therefore I_n + I_{n-1} = \displaystyle\int_0^1 u^{2n-2} du = \left[\frac{1}{2n-1}u^{2n-1}\right]_0^1 = \frac{1}{2n-1}$.

$\therefore I_n = \dfrac{1}{2n-1} - I_{n-1}$.

Q3cii $I_0 = \displaystyle\int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{4}$

$I_3 = \dfrac{1}{5} - I_2 = \dfrac{1}{5} - \left(\dfrac{1}{3} - I_1\right) = \dfrac{1}{5} - \left(\dfrac{1}{3} - (1 - I_0)\right) = \dfrac{1}{5} - \dfrac{1}{3} + 1 - \dfrac{\pi}{4}$

$= \dfrac{52 - 15\pi}{60}$.

Q3d Radius $r = \ell \sin\alpha$.
Speed of particle $P = v = \omega r = \omega\ell\sin\alpha$.
Vertically: $T\cos\alpha - mg = 0$ ……………………...(1)

Horizontally: $T\sin\alpha = \dfrac{mv^2}{r} = \dfrac{m\omega^2\ell^2\sin^2\alpha}{\ell\sin\alpha}$ ……(2)

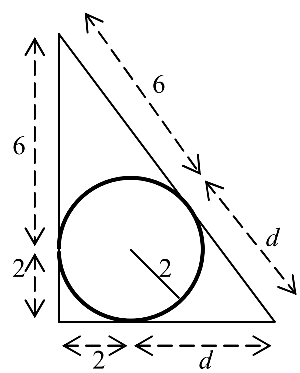
Solve (1) and (2) simultaneously, $\omega^2 = \dfrac{g}{\ell\cos\alpha}$.

Q4ai Area of $\Delta LOM = \dfrac{1}{2}kr$.

Q4aii Area of $\Delta KLM = A = \dfrac{1}{2}kr + \dfrac{1}{2}\ell r + \dfrac{1}{2}mr$

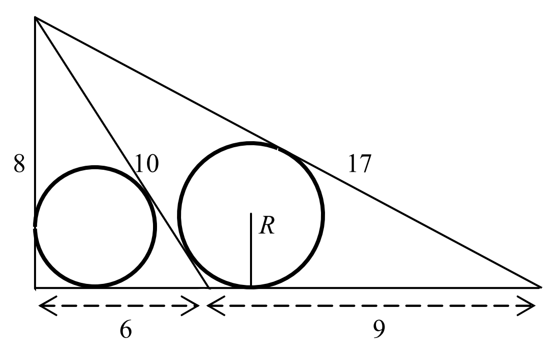
$= \dfrac{1}{2}(k + \ell + m)r = \dfrac{1}{2}P\,r$.

Q4aiii




$A = \dfrac{1}{2}P\,r$, $\dfrac{1}{2}(2+d)8 = \dfrac{1}{2}\left(8 + (2+d) + (6+d)\right)2$, $\therefore d = 4$.
$\therefore 2 + d = 6$ units from the foot of the fence.

Q4aiv Use Pythagoras theorem to find the side lengths.




$A = \dfrac{1}{2}P\,r$, $\dfrac{1}{2} \times 9 \times 8 = \dfrac{1}{2}(9 + 17 + 10)R$, $\therefore R = 2$ units.

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Q4bi $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

By implicit differentiation, $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$, $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$ .

Gradient of tangent at $P(x_1,y_1)$ is $m=-\frac{b^2x_1}{a^2y_1}$ .

Equation of tangent is $y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$.

Expand and rearrange to $\frac{x_1}{a^2}x+\frac{y_1}{b^2}y=\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}$ .

Since $P(x_1,y_1)$ is a point on the ellipse, $\therefore \frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}=1$ and the equation of tangent at $P(x_1,y_1)$ can be expressed as

$\frac{x_1}{a^2}x+\frac{y_1}{b^2}y=1$.

Q4bii Similarly, equation of tangent at $Q(x_2,y_2)$ can be expressed as $\frac{x_2}{a^2}x+\frac{y_2}{b^2}y=1$.

Let $T(u,v)$ be the intersection of the two tangents.

$\therefore \frac{x_1}{a^2}u+\frac{y_1}{b^2}v=1$ and $\frac{x_2}{a^2}u+\frac{y_2}{b^2}v=1$.

$\therefore \frac{x_1}{a^2}u+\frac{y_1}{b^2}v-\frac{x_2}{a^2}u-\frac{y_2}{b^2}v=0$.

Rearrange to $\frac{(x_1-x_2)}{a^2}u+\frac{(y_1-y_2)}{b^2}v=0$.

$\therefore T(u,v)$ lies on the line $\frac{(x_1-x_2)}{a^2}x+\frac{(y_1-y_2)}{b^2}y=0$.

Q4biii *M* is the midpoint of *PQ*, $\therefore M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$.

$\frac{(x_1-x_2)}{a^2}x+\frac{(y_1-y_2)}{b^2}y=0$ is a line through the origin *O*.

*M* satisfies the line because

$$\frac{(x_1-x_2)}{a^2}\times\frac{(x_1+x_2)}{2}+\frac{(y_1-y_2)}{b^2}\times\frac{(y_1+y_2)}{2}$$

$$=\frac{1}{2}\left(\frac{x_1^{\ 2}-x_2^{\ 2}}{a^2}+\frac{y_1^{\ 2}-y_2^{\ 2}}{b^2}\right)=\frac{1}{2}\left(\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}-\left(\frac{x_2^{\ 2}}{a^2}+\frac{y_2^{\ 2}}{b^2}\right)\right)$$

$$=\frac{1}{2}(1-1)=0.$$

$\therefore$*M* lies on the line *OT*, i.e. *O*, *M* and *T* are collinear.

Q5ai $P=\frac{21000}{7+3e^{-\frac{t}{3}}}$,

$$\frac{dP}{dt}=-\frac{21000}{\left(7+3e^{-\frac{t}{3}}\right)^2}\times\left(-e^{-\frac{t}{3}}\right)=\frac{21000e^{-\frac{t}{3}}}{\left(7+3e^{-\frac{t}{3}}\right)^2}.$$

$$\frac{1}{3}\left(1-\frac{P}{3000}\right)P=\frac{1}{3}\left(1-\frac{7}{7+3e^{-\frac{t}{3}}}\right)\frac{21000}{7+3e^{-\frac{t}{3}}}=\frac{1}{3}\left(\frac{3e^{-\frac{t}{3}}}{7+3e^{-\frac{t}{3}}}\right)\frac{21000}{7+3e^{-\frac{t}{3}}}$$

$$=\frac{21000}{\left(7+3e^{-\frac{t}{3}}\right)^2}.$$

$\therefore P$ satisfies $\frac{dP}{dt}=\frac{1}{3}\left(1-\frac{P}{3000}\right)P$.

Q5aii Today: $t=0$, $P=\frac{21000}{7+3e^{-\frac{t}{3}}}=\frac{21000}{7+3e^0}=2100$.

Q5aiii Eventually: $t\to\infty$, $e^{-\frac{t}{3}}\to 0$, $P\to\frac{21000}{7}=3000$.

Q5aiv Today: $t=0$, $P=2100$ and $\frac{dP}{dt}=\frac{1}{3}\left(1-\frac{P}{3000}\right)P$

$=\frac{1}{3}\left(1-\frac{2100}{3000}\right)2100=210$.

Annual % rate of growth today $=\frac{210}{2100}\times 100\%=10\%$.

Q5bi $p(x)=x^{n+1}-(n+1)x+n$, $p(1)=0$, $\therefore p(x)$ has a zero at $x=1$.

$p'(x)=(n+1)x^n-(n+1)$, $p'(1)=0$, $\therefore$ the zero at $x=1$ is a double zero.

Q5bii $p'(x)=(n+1)x^n-(n+1)$, $p''(x)=n(n+1)x^{n-1}\geq 0$ for $x\geq 0$. $\therefore p(x)$ is concave upward for $x\geq 0$. Also $p(x)$ touches the *x*-axis at $x=1$ (double zero), $\therefore p(x)\geq 0$ for $x\geq 0$.

Q5biii $p(x)=x^4-4x+3$ when $n=3$.

$p(x)$ has a double zero at $x=1$,

$\therefore p(x)=x^4-4x+3=(x-1)^2q(x)=\left(x^2-2x+1\right)\left(x^2+2x+3\right)$.

$\therefore p(x)=(x-1)^2\left(x^2+2x+3\right)$.

Q5ci $(x-a)^2=b^2-h^2$, $x=a\pm\sqrt{b^2-h^2}$.

$x_1=a-\sqrt{b^2-h^2}$ and $x_2=a+\sqrt{b^2-h^2}$.

Q5cii Area of cross-section $=\pi x_2^{\ 2}-\pi x_1^{\ 2}$

$$=\pi\left[\left(a+\sqrt{b^2-h^2}\right)^2-\left(a-\sqrt{b^2-h^2}\right)^2\right]=4\pi a\sqrt{b^2-h^2}.$$

Q5ciii Volume of torus:

$$V = 2\times\int_0^b 4\pi a\sqrt{b^2-h^2}\,dh = 8\pi a\int_0^b \sqrt{b^2-h^2}\,dh$$

$$= 8\pi a\times\frac{1}{4}\pi b^2 = 2\pi^2 ab^2 .$$

Note: $\int_0^r \sqrt{r^2-x^2}\,dx$ gives $\frac{1}{4}$ of the area of a circle of radius $r$.

Q6a Given $p(z)=z^3+az^2+bz+c$ has zeros 1, $-\omega$ and $-\overline{\omega}$, where $\omega^3=1$ and $\mathrm{Im}(\omega)>0$.

The roots of $z^3=1$ lie on the unit circle, and are separated by $\frac{2\pi}{3}$. $\therefore \omega = cis\frac{2\pi}{3}$ and $\overline{\omega}=cis\left(-\frac{2\pi}{3}\right)$.

Hence $\omega+\overline{\omega}=2\cos\frac{2\pi}{3}=-1$ and $\omega\overline{\omega}=cis0=1$.

$$\therefore z^3+az^2+bz+c=(z-1)(z+\omega)(z+\overline{\omega})$$

$$=(z-1)\left(z^2+(\omega+\overline{\omega})z+\omega\overline{\omega}\right)=(z-1)\left(z^2-z+1\right)$$

$$=z^3-2z^2+2z-1 .$$

Q6bi $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

By implicit differentiation, $\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$, $\frac{dy}{dx}=\frac{b^2x}{a^2y}$.

Gradient of tangent at $P(a\sec\theta, b\tan\theta)$:

$$m=\frac{dy}{dx}=\frac{b^2x}{a^2y}=\frac{b\sec\theta}{a\tan\theta}.$$

Equation of tangent: $y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$.

Expand and simplify: $bx\sec\theta-ay\tan\theta=ab\sec^2\theta-ab\tan^2\theta$

$=ab\left(\sec^2\theta-\tan^2\theta\right)=ab$.

$\therefore bx\sec\theta-ay\tan\theta-ab=0$.

Q6bii Shortest distance between line $ax+by+c=0$ and point $(u,v)$ is $\frac{|au+bv+c|}{\sqrt{a^2+b^2}}$.

$$\therefore SR=\frac{|abe\sec\theta-ab|}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}=\frac{ab|e\sec\theta-1|}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}.$$

$P(a\sec\theta, b\tan\theta)$ is on the right branch of the hyperbola ($e>1$),

$\therefore a\sec\theta\ge a$, $\sec\theta\ge 1$ and $SR=\frac{ab(e\sec\theta-1)}{\sqrt{a^2\tan^2\theta+b^2\sec^2\theta}}$.

Q6biii $S'R'=\frac{|-abe\sec\theta-ab|}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}=\frac{ab|-e\sec\theta-1|}{\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}$

$$=\frac{ab(e\sec\theta+1)}{\sqrt{a^2\tan^2\theta+b^2\sec^2\theta}}.$$

$$SR\times S'R'=\frac{ab(e\sec\theta-1)}{\sqrt{a^2\tan^2\theta+b^2\sec^2\theta}}\times\frac{ab(e\sec\theta+1)}{\sqrt{a^2\tan^2\theta+b^2\sec^2\theta}}$$

$$=\frac{a^2b^2\left(e^2\sec^2\theta-1\right)}{a^2\tan^2\theta+b^2\sec^2\theta}=\frac{a^2b^2\left(e^2\sec^2\theta-1\right)}{a^2\tan^2\theta+a^2\left(e^2-1\right)\sec^2\theta}$$

$$=\frac{a^2b^2\left(e^2\sec^2\theta-1\right)}{a^2\left(e^2\sec^2\theta+\tan^2\theta-\sec^2\theta\right)}$$

$$=\frac{a^2b^2\left(e^2\sec^2\theta-1\right)}{a^2\left(e^2\sec^2\theta-1\right)}=b^2 .$$

Q6ci

$$\frac{r}{r-1}\left[\frac{1}{{}^{n-1}C_{r-1}}-\frac{1}{{}^{n}C_{r-1}}\right]=\frac{r}{r-1}\left[\frac{(n-r)!(r-1)!}{(n-1)!}-\frac{(n-r+1)!(r-1)!}{n!}\right]$$

$$=\frac{r(r-1)!}{r-1}\left[\frac{n(n-r)!}{n(n-1)!}-\frac{(n-r+1)!}{n!}\right]=\frac{r!}{r-1}\left[\frac{n(n-r)!-(n-r+1)!}{n!}\right]$$

$$=\frac{r!}{r-1}\left[\frac{n(n-r)!-(n-r+1)(n-r)!}{n!}\right]=\frac{(n-r)!r!}{r-1}\left[\frac{n-(n-r+1)}{n!}\right]$$

$$=\frac{(n-r)!r!}{r-1}\left[\frac{r-1}{n!}\right]=\frac{(n-r)!r!}{n!}=\frac{1}{{}^{n}C_{r}}$$

Q6cii From Q6ci, $\frac{1}{{}^{n}C_{r}}=\frac{r}{r-1}\left[\frac{1}{{}^{n-1}C_{r-1}}-\frac{1}{{}^{n}C_{r-1}}\right]$.

$$\therefore \frac{1}{{}^{r}C_{r}}+\frac{1}{{}^{r+1}C_{r}}+\ldots+\frac{1}{{}^{m}C_{r}}$$

$$=\frac{r}{r-1}\left[\frac{1}{{}^{r-1}C_{r-1}}-\frac{1}{{}^{r}C_{r-1}}+\frac{1}{{}^{r}C_{r-1}}-\frac{1}{{}^{r+1}C_{r-1}}+\ldots+\frac{1}{{}^{m-1}C_{r-1}}-\frac{1}{{}^{m}C_{r-1}}\right]$$

$$=\frac{r}{r-1}\left[1-\frac{1}{{}^{m}C_{r-1}}\right].$$

Q6ciii As $m\to\infty$, $\frac{1}{{}^{m}C_{r-1}}\to 0$, $\sum_{n=r}^{m}\frac{1}{{}^{n}C_{r}}\to\frac{r}{r-1}$.

Q7ai $p_s=3\times\frac{{}^{n}C_3\times{}^{n}C_0\times{}^{n}C_0}{{}^{3n}C_3}=3\times\frac{{}^{n}C_3}{{}^{3n}C_3}$.

Q7aii $p_d=\frac{{}^{n}C_1\times{}^{n}C_1\times{}^{n}C_1}{{}^{3n}C_3}=\frac{\left({}^{n}C_1\right)^3}{{}^{3n}C_3}$.

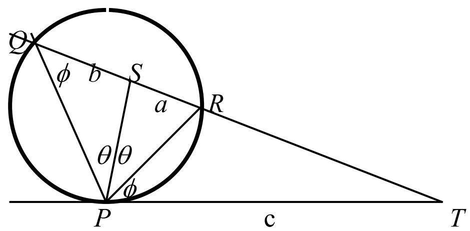
Q7aiii $p_m=3!\times\frac{{}^{n}C_2\times{}^{n}C_1\times{}^{n}C_0}{{}^{3n}C_3}=6\times\frac{{}^{n}C_2\times{}^{n}C_1}{{}^{3n}C_3}$.

Q7aiv $p_s:p_d:p_m=3\times{}^{n}C_3:\left({}^{n}C_1\right)^3:6\times{}^{n}C_2\times{}^{n}C_1$

$$=3\times\frac{n(n-1)(n-2)}{3!}:n^3:6\times\frac{n(n-1)}{2!}\times n$$

$\approx\frac{n^3}{2}:n^3:3n^3=1:2:6$ for large $n$.

Q7bi




Let $\angle PQR = \phi$.

$\angle TSP = \theta + \phi$ (Exterior angle equals sum of opposite interior angles).

$\angle RPT = \phi$ (Equal angles in alternate segments).

$\angle TPS = \theta + \phi$ (See diagram).

$\therefore \angle TSP = \angle TPS$.

Q7bii $TS = TP = c$ ($\Delta TSP$ is isosceles), $\therefore RT = c - a$ and $QT = b + c$.

$\angle TRP = 2\theta + \phi$ (Exterior angle equals sum of opposite interior angles).

$\angle TPQ = 2\theta + \phi$ (See diagram).

$\therefore \angle TRP = \angle TPQ$. $\therefore \Delta TRP$ and $\Delta TPQ$ are similar.

Hence $\dfrac{c}{c-a} = \dfrac{b+c}{c}$, $c^2 = (b+c)(c-a)$.

Expand and simplify, $bc = ca + ab$.

Both sides divided by $abc$, $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$.

Q7ci $v = b - (b - v_0)e^{-\alpha t}$, $v_0$, $b$ and $\alpha \in R^+$, and $v_0 < b$.

$\dfrac{dv}{dt} = \alpha(b - v_0)e^{-\alpha t} = \alpha(b - v)$.

Q7cii $b$ represents the velocity of the current.

Q7ciii $v = \dfrac{dx}{dt} = b - (b - v_0)e^{-\alpha t}$,

$x = \int \left[ b - (b - v_0)e^{-\alpha t} \right] dt = bt + \dfrac{(b - v_0)e^{-\alpha t}}{\alpha} + c$.

At the start of the drift, $t = 0$, $x = 0$, $\therefore c = -\dfrac{b - v_0}{\alpha}$, and

$x = bt + \dfrac{b - v_0}{\alpha}\left(e^{-\alpha t} - 1\right)$.

Given $v = b - (b - v_0)e^{-\alpha t}$, $\therefore e^{-\alpha t} = \dfrac{b - v}{b - v_0}$, $t = \dfrac{1}{\alpha}\log_e\left(\dfrac{b - v_0}{b - v}\right)$

and $e^{-\alpha t} - 1 = \dfrac{v_0 - v}{b - v_0}$.

$\therefore x = \dfrac{b}{\alpha}\log_e\left(\dfrac{b - v_0}{b - v}\right) + \dfrac{v_0 - v}{\alpha}$.

Q7civ Given $v_0 = \dfrac{b}{10} = 0.1b$, when $v = \dfrac{b}{2} = 0.5b$,

$x = \dfrac{b}{\alpha}\log_e\left(\dfrac{b - 0.1b}{b - 0.5b}\right) + \dfrac{0.1b - 0.5b}{\alpha} = \dfrac{b}{\alpha}\left(\log_e 1.8 - 0.4\right) \approx 0.19\dfrac{b}{\alpha}$.

Q8a The statement: $\cos\theta + \cos 3\theta + ... + \cos(2n-1)\theta = \dfrac{\sin 2n\theta}{2\sin\theta}$.

When $n = 1$, $RHS = \dfrac{\sin 2\theta}{2\sin\theta} = \dfrac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta = LHS$. $\therefore$ the statement is true when $n = 1$.

Assume it is true when $n = k$,

i.e. $\cos\theta + \cos 3\theta + ... + \cos(2k-1)\theta = \dfrac{\sin 2k\theta}{2\sin\theta}$.

When $n = k + 1$,

$LHS = \cos\theta + \cos 3\theta + ... + \cos(2k-1)\theta + \cos(2(k+1)-1)$

$= \cos\theta + \cos 3\theta + ... + \cos(2k-1)\theta + \cos(2k+1)\theta$,

$RHS = \dfrac{\sin 2(k+1)\theta}{2\sin\theta} = \dfrac{\sin(2k\theta + 2\theta)}{2\sin\theta}$

$= \dfrac{\sin 2k\theta\cos 2\theta + \cos 2k\theta\sin 2\theta}{2\sin\theta}$

$= \dfrac{\sin 2k\theta\left(1 - 2\sin^2\theta\right) + 2\cos 2k\theta\sin\theta\cos\theta}{2\sin\theta}$

$= \dfrac{\sin 2k\theta}{2\sin\theta} + \dfrac{2\cos 2k\theta\sin\theta\cos\theta - 2\sin 2k\theta\sin^2\theta}{2\sin\theta}$

$= \dfrac{\sin 2k\theta}{2\sin\theta} + \cos 2k\theta\cos\theta - \sin 2k\theta\sin\theta$

$= \dfrac{\sin 2k\theta}{2\sin\theta} + \cos(2k\theta + \theta)$

$= \cos\theta + \cos 3\theta + ... + \cos(2k-1)\theta + \cos(2k+1)\theta = LHS$.

$\therefore$ the statement is true for $n = k + 1$.

Hence it is true for $n \ge 1$.

Q8bi

$A = \sum_{k=1}^{n} A_k = 2\pi R^2 \sin\delta\left[\cos\dfrac{\delta}{2} + \cos\dfrac{3\delta}{2} + ... + \cos\dfrac{(2n-1)\delta}{2}\right]$

$= 2\pi R^2 \sin\delta\left[\dfrac{\sin n\delta}{2\sin\frac{\delta}{2}}\right] = 2\pi R^2\left(2\sin\dfrac{\delta}{2}\cos\dfrac{\delta}{2}\right)\left[\dfrac{\sin n\delta}{2\sin\frac{\delta}{2}}\right]$

$= 2\pi R^2 \cos\dfrac{\delta}{2}\sin n\delta$.

Since $n\delta = \dfrac{\pi}{2}$, $\therefore \dfrac{\delta}{2} = \dfrac{\pi}{4n}$, and hence $A = 2\pi R^2 \cos\dfrac{\pi}{4n}$.

Q8bii As $n \to \infty$, $\cos\dfrac{\pi}{4n} \to 1$, $A \to 2\pi R^2$ ($\frac{1}{2}$ of the surface area of a sphere).

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Q8ci $f(t)=\sin(a+nt)\sin b-\sin a\sin(b-nt)$, where $a>0$, $b>0$, $a+b<\pi$ and $n\neq 0$.

$f(0)=\sin(a+0)\sin b-\sin a\sin(b-0)=0$.

$f'(t)=n\cos(a+nt)\sin b+n\sin a\cos(b-nt)$

$f''(t)=-n^2\sin(a+nt)\sin b-n^2\sin a\sin(b-nt)$

$=-n^2(\sin(a+nt)\sin b+\sin a\sin(b-nt))$

$\therefore f''(t)=-n^2 f(t)$.

Q8cii

$f(t)=(\sin a\cos nt+\cos a\sin nt)\sin b-\sin a(\sin b\cos nt-\cos b\sin nt)$

$=\sin a\cos nt\sin b+\cos a\sin nt\sin b-\sin a\sin b\cos nt+\sin a\cos b\sin nt$

$=\cos a\sin nt\sin b+\sin a\cos b\sin nt$

$=(\sin a\cos b+\cos a\sin b)\sin nt$

$=\sin(a+b)\sin nt$.

Q8ciii $\dfrac{\sin(a+nt)}{\sin(b-nt)}=\dfrac{\sin a}{\sin b}$, $\sin(b-nt)\neq 0$ and $\sin b\neq 0$.

$\therefore \sin(a+nt)\sin b-\sin a\sin(b-nt)=0$,

i.e. $\sin(a+b)\sin nt=0$.

Since $0<a+b<\pi$, $\sin(a+b)\neq 0$,

$\therefore \sin nt=0$ and $\sin(b-nt)\neq 0$.

Hence $nt=k\pi$, $\therefore t=\dfrac{k\pi}{n}$ where $k\in J$.

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