NSW BOS Mathematics Extension 1 Solutions 2006
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Q1a $\int \frac{dx}{49+x^2} = \frac{1}{7}\int \frac{7}{7^2+x^2}dx = \frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)+c$.

Q1b $u = x^4+8$, $\frac{du}{dx} = 4x^3$.

$\int x^3\sqrt{x^4+8}\,dx = \int \frac{1}{4}\sqrt{u}\frac{du}{dx}dx = \int \frac{1}{4}u^{\frac{1}{2}}du = \frac{u^{\frac{3}{2}}}{6}+c = \frac{\left(x^4+8\right)^{\frac{3}{2}}}{6}+c$

Q1c $\lim_{x\to 0}\frac{\sin 5x}{3x} = \lim_{x\to 0}\frac{5}{3}\times\frac{\sin 5x}{5x} = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}$.

Q1d
$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}-1 = \frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{(\sin\theta+\cos\theta)}-1$

$= (\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)-1 = (1-\sin\theta\cos\theta)-1$

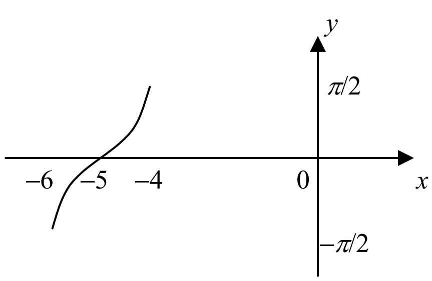
$= -\sin\theta\cos\theta = -\frac{1}{2}\sin 2\theta$.

Q1e $y = x^3$, $\frac{dy}{dx} = 3x^2 = 12$, $\therefore x = \pm 2$ and $y = \pm 8$.
$y = 12x+b$, $8 = 12(2)+b$, $b = -16$.
$-8 = 12(-2)+b$, $b = 16$.

Q2ai $f(x) = \sin^{-1}(x+5)$ is defined for $-1 \le x+5 \le 1$,
i.e. $-6 \le x \le -4$. Domain: $[-6,-4]$, range: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Q2aii $f'(x) = \frac{1}{\sqrt{1-(x+5)^2}}$, $f'(-5) = \frac{1}{\sqrt{1-(-5+5)^2}} = 1$.

Q2aiii



Q2bi $(1+x)^n = 1+\binom{n}{1}x+\binom{n}{2}x^2+......+\binom{n}{r}x^r+......+\binom{n}{n}x^n$,

$\frac{d}{dx}(1+x)^n = \frac{d}{dx}\left[1+\binom{n}{1}x+\binom{n}{2}x^2+......+\binom{n}{r}x^r+......+\binom{n}{n}x^n\right]$

$n(1+x)^{n-1} = \binom{n}{1}+2\binom{n}{2}x+......+r\binom{n}{r}x^{r-1}+......+n\binom{n}{n}x^{n-1}$.

Q2bii Let $x = 2$,
$n3^{n-1} = \binom{n}{1}+......+r\binom{n}{r}2^{r-1}+......+n\binom{n}{n}2^{n-1} = \sum_{r=1}^{n} r\binom{n}{r}2^{r-1}$.

Q2ci Given $y = \frac{1}{2}(p+r)x-apr$. At point U, $x = 0$,
$\therefore y = -apr$, hence $U(0,-apr)$.

Q2cii Given the tangent at $P(2ap,ap^2)$ is $y = px-ap^2$, by symmetry the tangent at $Q(2aq,aq^2)$ is $y = qx-aq^2$.
At point T, $px-ap^2 = qx-aq^2$, $\therefore px-qx = ap^2-aq^2$,
$(p-q)x = a(p-q)(p+q)$, $\therefore x = a(p+q)$, where $p \ne q$.
$\therefore y = px-ap^2 = pa(p+q)-ap^2 = apq$.
Hence $T(a(p+q),apq)$.

Q2ciii Since $Q(2aq,aq^2)$ and $R(2ar,ar^2)$ are two different points having the same y-coordinate, $\therefore q^2 = r^2$ and $q \ne r$,
$\therefore q = -r$.
$\therefore U(0,-apr)$ and $T(a(p+q),apq)$ are two different points having the same y-coordinate, $\therefore TU$ is a horizontal line perpendicular to the vertical axis of the parabola.

Q3a $\int_0^{\frac{\pi}{4}}\sin^2 x\,dx = \int_0^{\frac{\pi}{4}}\frac{1}{2}(1-\cos 2x)dx = \left[\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)\right]_0^{\frac{\pi}{4}}$
$= \frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi-2}{8}$.

Q3bi $f(x) = 3\log_e x-x$, $f(1.5) = 3\log_e 1.5-1.5 = -0.2836$,
$f(2) = 3\log_e 2-2 = 0.0794$. $\therefore f(x)$ has a zero at $1.5 < x < 2$,
$\therefore x$-coordinate of P is $1.5 < x < 2$.

Q3bii $f'(x) = \frac{3}{x}-1$, $x_1 = 1.5$, $f'(1.5) = \frac{3}{1.5}-1 = 1$.
$x_2 = x_1-\frac{f(x_1)}{f'(x_1)} = 1.5-\frac{-0.2836}{1} = 1.78$.

Q3ci Three blocks high: $5\times 4\times 3 = 60$

Q3cii Total number of 2, 3, 4 and 5-block towers:
$5\times 4+5\times 4\times 3+5\times 4\times 3\times 2+5\times 4\times 3\times 2\times 1 = 320$

Q3di Since $\angle QKT+\angle QMT = 180^0$, $\therefore QKTM$ is a cyclic quadrilateral.

Q3dii $\angle KMT$ and $\angle KQT$ are subtended by the same arc KT, $\therefore$ they are equal.

Q3diii $\angle PTN = \angle PQT$ (The angle between a chord and a tangent at an end of the chord equals the angle in the alternate segment).
$\angle PTN = \angle PQT = \angle KQT = \angle KMT$, i.e. $\angle PTN$ and $\angle KMT$ are equal corresponding angles, $\therefore MK // TP$.

Q4ai and Q4aii $P(x)=x^3+rx^2+sx+t=(x-1)(x-\alpha)(x+\alpha)$
$=x^3-x^2-\alpha^2x+\alpha^2$.
Comparing coefficients: $r=-1$, $s=-\alpha^2$, $t=\alpha^2$.
$\therefore s+t=0$.

Q4bi $T=\frac{2\pi}{n}$, $5=\frac{2\pi}{n}$, $\therefore n=\frac{2\pi}{5}$. $\therefore x=18\cos\left(\frac{2\pi}{5}t\right)$.

Q4bii A rest position is at one of the extreme positions.
First time at $x=9$, $9=18\cos\left(\frac{2\pi}{5}t\right)$, $\therefore \cos\left(\frac{2\pi}{5}t\right)=0.5$,
$\frac{2\pi}{5}t=\frac{\pi}{3}$, $t=\frac{5}{6}$ s. Required time is $\frac{5}{6}$ s.

Q4ci $\ddot{x}=18x^3+27x^2+9x$. At $t=0$, $x=-2$, $v=-6$.
$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=18x^3+27x^2+9x$,
$\frac{1}{2}v^2=\int\left(18x^3+27x^2+9x\right)dx$, $\frac{1}{2}v^2=\frac{9x^4}{2}+9x^3+\frac{9x^2}{2}+d$,

$\therefore v^2=9x^4+18x^3+9x^2+c$.
Using $x=-2$, $v=-6$, c is 0.
$\therefore v^2=9x^4+18x^3+9x^2=9x^2\left(x^2+2x+1\right)=9x^2(x+1)^2$.

Q4cii At $t=0$, $x=-2$, $v=-6$, $\ddot{x}=-54$, $\therefore v$ remains negative and $x<-2$ as time progresses.
$\therefore v=-\sqrt{9x^2(x+1)^2}=-3x(1+x)$, i.e. $\frac{dx}{dt}=-3x(1+x)$.
Hence $\frac{dt}{dx}=-\frac{1}{3}\times\frac{1}{x(1+x)}$, $t=-\frac{1}{3}\int\frac{1}{x(1+x)}dx$,
$\therefore \int\frac{1}{x(1+x)}dx=-3t$.

Q4ciii $\log_e\left(1+\frac{1}{x}\right)=3t+c$. At $t=0$, $x=-2$,
$\therefore \log_e\left(1-\frac{1}{2}\right)=c$, $c=\log_e\frac{1}{2}=-\log_e 2$.
$\therefore \log_e\left(1+\frac{1}{x}\right)=3t-\log_e 2$, $\log_e\left(1+\frac{1}{x}\right)+\log_e 2=3t$,
$\therefore \log_e 2\left(1+\frac{1}{x}\right)=3t$, $2\left(1+\frac{1}{x}\right)=e^{3t}$, $1+\frac{1}{x}=0.5e^{3t}$,
$\frac{1}{x}=0.5e^{3t}-1$, $x=\frac{1}{0.5e^{3t}-1}$.

Q5a $y=10e^{-0.7t}+3$, $\frac{dy}{dt}=-0.7\left(10e^{-0.7t}\right)=-0.7(y-3)$.

Q5b All functions (relations) have an inverse. Does the question mean to show that $f(x)=\log_e\left(1+e^x\right)$ has an inverse **function**?
$f'(x)=\frac{e^x}{1+e^x}>0$ for all x, $\therefore f(x)$ is an increasing function for all x, $\therefore f(x)$ is a one-to-one function and hence $f^{-1}(x)$ exists.

Q5ci $V=\frac{\pi}{3}x^2(3r-x)=\pi rx^2-\frac{\pi}{3}x^3$, $\frac{dV}{dx}=2\pi rx-\pi x^2$.
$\frac{dV}{dt}=\frac{dV}{dx}\times\frac{dx}{dt}$, $\therefore \frac{dx}{dt}=\frac{\frac{dV}{dt}}{\frac{dV}{dx}}=\frac{k}{2\pi rx-\pi x^2}=\frac{k}{\pi x(2r-x)}$.

Q5cii Since the rate is constant, $V=kt$, $t=\frac{V}{k}$.
When $x=\frac{1}{3}r$, $t_i=\frac{8\pi}{81k}r^3$.
When $x=\frac{2}{3}r$, $t_f=\frac{28\pi}{81k}r^3$.
$\therefore \frac{t_f}{t_i}=\frac{28}{8}=3.5$

Q5di $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$,
$\therefore 1+\tan\alpha\tan\beta=\frac{\tan\alpha-\tan\beta}{\tan(\alpha-\beta)}$.
Let $\alpha=(n+1)\theta$ and $\beta=n\theta$.
$1+\tan(n+1)\theta\tan n\theta=\frac{\tan(n+1)\theta-\tan n\theta}{\tan\theta}$,
$\therefore 1+\tan(n+1)\theta\tan n\theta=\cot\theta(\tan(n+1)\theta-\tan n\theta)$,
$\therefore 1+\tan n\theta\tan(n+1)\theta=\cot\theta(\tan(n+1)\theta-\tan n\theta)$.

Q5dii Given $1+\tan n\theta\tan(n+1)\theta=\cot\theta(\tan(n+1)\theta-\tan n\theta)$, prove by induction that for $n\ge 1$,
$\tan\theta\tan 2\theta+\tan 2\theta\tan 3\theta+...+\tan n\theta\tan(n+1)\theta$
$=-(n+1)+\cot\theta\tan(n+1)\theta$

It is true for $n=1$: $1+\tan\theta\tan 2\theta=\cot\theta(\tan 2\theta-\tan\theta)$,
$1+\tan\theta\tan 2\theta=\cot\theta\tan 2\theta-\cot\theta\tan\theta$,
$\therefore \tan\theta\tan 2\theta=-2+\cot\theta\tan 2\theta$.

Assume that it is true for $n=k$, it is also true for $n=k+1$:
$\tan\theta\tan 2\theta+\tan 2\theta\tan 3\theta+...+\tan k\theta\tan(k+1)\theta$
$=-(k+1)+\cot\theta\tan(k+1)\theta$
then $\tan\theta\tan 2\theta+\tan 2\theta\tan 3\theta+...+\tan k\theta\tan(k+1)\theta+\tan(k+1)\theta\tan(k+2)\theta$
$=-(k+1)+\cot\theta\tan(k+1)\theta+\tan(k+1)\theta\tan(k+2)\theta$
$=-(k+1)+\cot\theta\tan(k+1)\theta+\cot\theta(\tan(k+2)\theta-\tan(k+1)\theta)-1$
$=-((k+1)+1)+\cot\theta\tan((k+1)+1)\theta$.
$\therefore$it is true for all $n\ge 1$.

Q6ai $L^2=(x_2-x_1)^2+(y_2-y_1)^2$
$=(a-Vt\cos\theta)^2+(Vt-Vt\sin\theta)^2$
$=a^2-2aVt\cos\theta+V^2t^2\cos^2\theta+V^2t^2-2V^2t^2\sin\theta+V^2t^2\sin^2\theta$
$=V^2t^2\sin^2\theta+V^2t^2\cos^2\theta+V^2t^2-2V^2t^2\sin\theta-2aVt\cos\theta+a^2$
$=V^2t^2\left(\sin^2\theta+\cos^2\theta\right)+V^2t^2-2V^2t^2\sin\theta-2aVt\cos\theta+a^2$
$=2V^2t^2(1-\sin\theta)-2aVt\cos\theta+a^2$

Q6aii Smallest distance occurs at where $\frac{d(L^2)}{dt} = 0$,

$4V^2(1-\sin\theta)t - 2aV\cos\theta = 0$, $\therefore t = \frac{a\cos\theta}{2V(1-\sin\theta)}$.

$L = \sqrt{2V^2t^2(1-\sin\theta) - 2aVt\cos\theta + a^2}$,

$$L_{\min} = \sqrt{2V^2\left(\frac{a\cos\theta}{2V(1-\sin\theta)}\right)^2(1-\sin\theta) - 2aV\frac{a\cos\theta}{2V(1-\sin\theta)}\cos\theta + a^2}$$

$$= \sqrt{\frac{a^2\cos^2\theta}{2(1-\sin\theta)} - \frac{a^2\cos^2\theta}{(1-\sin\theta)} + a^2}$$

$$= a\sqrt{1 - \frac{\cos^2\theta}{2(1-\sin\theta)}} = a\sqrt{1 - \frac{1-\sin^2\theta}{2(1-\sin\theta)}}$$

$$= a\sqrt{1 - \frac{(1-\sin\theta)(1+\sin\theta)}{2(1-\sin\theta)}} = a\sqrt{1 - \frac{1+\sin\theta}{2}} = a\sqrt{\frac{1-\sin\theta}{2}}.$$

Q6aiii Particle 1 is ascending when its $\frac{dy}{dt} > 0$,

i.e. $V\sin\theta - gt > 0$, $t < \frac{V\sin\theta}{g}$.

$\therefore \frac{a\cos\theta}{2V(1-\sin\theta)} < \frac{V\sin\theta}{g}$, $\therefore V^2 > \frac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}$,

$\therefore V > \sqrt{\frac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}}$.

Q6bi Four-member team: Let Y be the number of competitors not completing the course.

$\Pr(Y \ge 3) = \Pr(Y = 3) + \Pr(Y = 4) = \binom{4}{3}pq^3 + q^4 = 4pq^3 + q^4$.

Q6bii Four-member team. Let X be the number of competitors completing the course.

$\Pr(X \ge 2) = \binom{4}{2}p^2q^2 + \binom{4}{3}p^3q + p^4$

$= 6(1-q)^2q^2 + 4(1-q)^3q + (1-q)^4$

$= (1-q)^2\left(6q^2 + 4q(1-q) + (1-q)^2\right)$

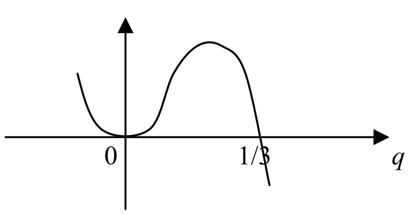
$= (1-q)^2(1 + 2q + 3q^2)$.

Q6biii Two-member team: $\Pr(X \ge 1) = 1 - \Pr(X = 0) = 1 - q^2$.

Q6biv $1 - q^2 > (1-q)^2(1 + 2q + 3q^2)$,

$(1-q)(1+q) > (1-q)^2(1 + 2q + 3q^2)$,

$1 + q > (1-q)(1 + 2q + 3q^2)$, $\therefore q^2 - 3q^3 < 0$, $q^2(1-3q) < 0$.



$q^2(1-3q) < 0$ when $q > \frac{1}{3}$, $\therefore \frac{1}{3} < q < 1$.

Q7a $A = \frac{1}{2}r^2(2\theta) - \frac{1}{2}r^2\sin 2\theta = r^2\theta - \frac{1}{2}r^2(2\sin\theta\cos\theta)$

$= r^2(\theta - \sin\theta\cos\theta)$.

Q7b $w = r(2\theta)$, $r = \frac{w}{2\theta}$, $\therefore A = \frac{w^2}{4\theta^2}(\theta - \sin\theta\cos\theta)$.

$\frac{dA}{d\theta} = -\frac{2w^2}{4\theta^3}(\theta - \sin\theta\cos\theta) + \frac{w^2}{4\theta^2}(1 - \cos^2\theta + \sin^2\theta)$

$= -\frac{2w^2}{4\theta^3}(\theta - \sin\theta\cos\theta) + \frac{w^2}{4\theta^2}(1 - \cos^2\theta + 1 - \cos^2\theta)$

$= -\frac{2w^2}{4\theta^3}(\theta - \sin\theta\cos\theta) + \frac{2w^2}{4\theta^2}(1 - \cos^2\theta)$

$= \frac{w^2}{2\theta^3}(\sin\theta\cos\theta - \theta\cos^2\theta)$

$= \frac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}$.

Q7c Let $g(\theta) = \sin\theta - \theta\cos\theta$,

$g'(\theta) = \cos\theta - \cos\theta + \theta\sin\theta = \theta\sin\theta$.

For $0 < \theta < \pi$, $g'(\theta) = \theta\sin\theta > 0$,

$\therefore g(\theta) = \sin\theta - \theta\cos\theta$ is an increasing function in $0 < \theta < \pi$,

and $g(0) = 0$, hence $g(\theta) = \sin\theta - \theta\cos\theta > 0$ for $0 < \theta < \pi$.

Q7d $\frac{dA}{d\theta} = 0$, $\frac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3} = 0$, where $0 < \theta < \pi$.

Since $w > 0$ and $\sin\theta - \theta\cos\theta > 0$,

$\therefore \cos\theta = 0$ and $\theta = \frac{\pi}{2}$ is the only solution in $0 < \theta < \pi$.

Q7e

θ	1	$\frac{\pi}{2}$	2
$\frac{dA}{d\theta}$	+	0	–

$\therefore A$ is maximum when $\theta = \frac{\pi}{2}$.

$\therefore A = \frac{w^2}{4\theta^2}(\theta - \sin\theta\cos\theta)$,

$A_{\max} = \frac{w^2}{4\left(\frac{\pi}{2}\right)^2}\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\cos\frac{\pi}{2}\right) = \frac{w^2}{2\pi}$ square units.

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