NSW BOS Mathematics Extension 1 Solutions 2007
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Q1a $\left(1+\sqrt{5}\right)^3 = 1^3 + 3(1)^2\left(\sqrt{5}\right) + 3(1)\left(\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^3 = 16 + 8\sqrt{5}$.

Q1b $x = \dfrac{(4)(3)+(19)(2)}{2+3} = 10$; $y = \dfrac{(5)(3)+(-5)(2)}{2+3} = 1$.

Q1c $\dfrac{d}{dx}\left(\tan^{-1}\left(x^4\right)\right) = \left(\dfrac{d}{d\left(x^4\right)}\left(\tan^{-1}\left(x^4\right)\right)\right)\left(\dfrac{d}{dx}\left(x^4\right)\right)$

$= \left(\dfrac{1}{1+\left(x^4\right)^2}\right)\left(4x^3\right) = \dfrac{4x^3}{1+x^8}$.

Q1d Line 1: $x - 2y + 3 = 0$, $y = \dfrac{1}{2}x + \dfrac{3}{2}$, $\dfrac{dy}{dx} = \dfrac{1}{2}$, $m_1 = \dfrac{1}{2}$.

Curve 2: $y = x^3 + 1$, $\dfrac{dy}{dx} = 3x^2$. At $x = 1$, $\dfrac{dy}{dx} = 3$, $m_2 = 3$.

$\tan\theta = \dfrac{m_2 - m_1}{1 + m_2 m_1} = \dfrac{5/2}{1+3/2} = 1$, $\therefore \theta = \dfrac{\pi}{4}$.

Q1e Given $u = 25 - x^2$, $\therefore -\dfrac{du}{dx} = 2x$.

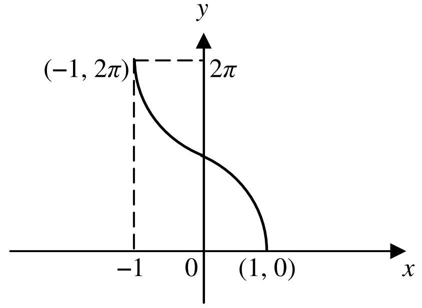
When $x = 3$, $u = 16$; when $x = 4$, $u = 9$.

$\displaystyle\int_3^4 \frac{2x}{\sqrt{25-x^2}}dx = -\int_3^4 \frac{1}{\sqrt{u}}\frac{du}{dx}dx = -\int_{16}^{9} u^{-\frac{1}{2}}du = \int_9^{16} u^{-\frac{1}{2}}du$

$= \left[2u^{\frac{1}{2}}\right]_9^{16} = 2$.

Q2a $LHS = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{2\sin^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}} = \dfrac{\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}} = \tan\dfrac{\theta}{2} = RHS$.

Q2bi



Q2bii The range is $[0, 2\pi]$.

Q2c $P(x) = x^2 + ax + b$,

$P(2) = 4 + 2a + b = 0$, $\therefore 2a + b = -4$.

$P(-1) = 1 - a + b = 18$, $\therefore a - b = -17$.

Solve simultaneously, $a = -7$ and $b = 10$.

Q2di $v = 50\left(1 - e^{-0.2t}\right)$, $a = \dfrac{dv}{dt} = 50\left(0.2e^{-0.2t}\right) = 10e^{-0.2t}$.

When $t = 10$, $a = 10e^{-2} \approx 1.4\ \text{ms}^{-2}$.

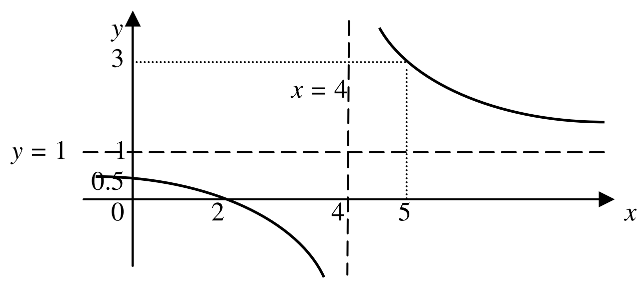
Q2dii Displacement in the first 10 seconds

$= \displaystyle\int_0^{10} 50\left(1 - e^{-0.2t}\right)dt = \left[50\left(t + 5e^{-0.2t}\right)\right]_0^{10} = 50\left(10 + 5e^{-2} - 5\right) \approx 284$.

Distance fallen = 284 m.

Q3a $V = \displaystyle\int_0^3 \pi y^2 dx = \int_0^3 \frac{\pi}{9+x^2}dx = \left[\frac{\pi}{3}\tan^{-1}\frac{x}{3}\right]_0^3 = \frac{\pi}{3}\tan^{-1}1 = \frac{\pi^2}{12}$

Q3bi $y = \dfrac{x-2}{x-4} = 1 - \dfrac{2}{x-4}$. Vertical asymptote: $x = 4$; horizontal asymptote: $y = 1$.



Q3bii $\dfrac{x-2}{x-4} = 3$ when $x - 2 = 3x - 12$, i.e. $x = 5$.

From graph, $\dfrac{x-2}{x-4} \le 3$ when $x < 4$ or $x \ge 5$.

Q3ci $\ddot{x} = -e^{-2x}$, $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = -e^{-2x}$,

$\dfrac{1}{2}v^2 = \displaystyle\int\left(-e^{-2x}\right)dx = \frac{1}{2}e^{-2x} + c_1$.

Given $x = 0$ and $v = 1$ when $t = 0$,

$\therefore c_1 = 0$, $\therefore v^2 = e^{-2x}$, $\therefore \dot{x} = v = e^{-x}$. Note: $\dot{x} = -e^{-x}$ does not meet the initial conditions.

Q3cii $\dot{x} = e^{-x}$, $\therefore \dfrac{dx}{dt} = e^{-x}$, $\dfrac{dt}{dx} = e^x$, $\therefore t = \displaystyle\int e^x dx = e^x + c_2$.

Given $x = 0$ and $v = 1$ when $t = 0$,

$\therefore c_2 = -1$ and $t = e^x - 1$. Hence $x = \log_e(t+1)$.

Q4ai $\Pr(X = 2) = 0.1^2 = 0.01$

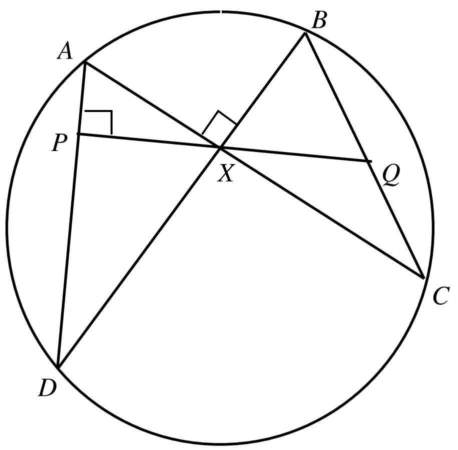
Q4aii $\Pr(X = 2) = {}^{20}C_2(0.1)^2(0.9)^{18} \approx 0.285$

Q4aiii $\Pr(X > 2) = 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2)$

$= 1 - 0.9^{20} - {}^{20}C_1(0.1)(0.9)^{19} - 0.285 \approx 0.32$.

Q4b For $n = 1$, $7^{2n-1} + 5 = 12$ is divisible by 12.
For $n = k$, assume that $7^{2k-1} + 5$ is divisible by 12.
For $n = k + 1$,
$7^{2(k+1)-1} + 5 = 7^2 7^{2k-1} + 5 = 49\left(7^{2k-1} + 5\right) - 48 \times 5$ is divisible by 12 because both $7^{2k-1} + 5$ and 48 are divisible by 12.
Hence $7^{2n-1} + 5 = 12$ is divisible by 12 for all integers $n \ge 1$.

Q4ci



$\angle QBX = \angle XAP$ (angles subtended on the circumference by the same arc).
$\angle XAP + \angle AXP = \angle PXD + \angle AXP = 90°$, $\therefore \angle XAP = \angle PXD$.
$\angle PXD = \angle QXB$ (vertically opposite angles).
$\therefore \angle QXB = \angle QBX$.

Q4cii $\angle QXB = \angle QBX$, $\therefore \Delta QXB$ is isosceles and $QB = QX$.
Similarly, $\angle QXC = \angle QCX$, $\therefore \Delta QXC$ is isosceles and $QX = QC$. $\therefore QB = QC$ and hence Q bisects BC.

Q5ai Area of sector $OPQ = \frac{1}{2} r^2 \theta$.

Area of $\Delta OPT = \frac{1}{2} r(PT) = \frac{1}{2} r(r \tan\theta) = \frac{1}{2} r^2 \tan\theta$.

$\therefore \frac{1}{2} r^2 \tan\theta = 2 \times \frac{1}{2} r^2 \theta$, $\therefore \tan\theta = 2\theta$.

Q5aii Let $f(\theta) = 2\theta - \tan\theta$. $f'(\theta) = 2 - \sec^2\theta$.
Newton's method:
$\theta_1 \approx \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)} = 1.15 - \frac{2 \times 1.15 - \tan 1.15}{2 - \sec^2 1.15} = 1.1664$.

Q5b There are 4! permutations of the four children together, and there are 3! ways in arranging a particular permutation of the four children, Mr Roberts and Mrs Roberts.
$\therefore$ total number of arrangements of the family of six with the children together is 3!4! .
Total number of arrangements of the family of six without restriction is 6!.
$\therefore \Pr(children.together) = \frac{3!4!}{6!} = \frac{1}{5}$.

Q5c $\sin^{-1} x + \frac{1}{2}\cos^{-1} y = \frac{\pi}{3}$ (1)

$3\sin^{-1} x - \frac{1}{2}\cos^{-1} y = \frac{2\pi}{3}$ (2)

(1) + (2): $4\sin^{-1} x = \pi$, $\sin^{-1} x = \frac{\pi}{4}$, $\therefore x = \frac{1}{\sqrt{2}}$.

From (1), $\cos^{-1} y = 2\left(\frac{\pi}{3} - \sin^{-1} x\right) = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$,

$\therefore y = \frac{\sqrt{3}}{2}$.

Q5di Substitute $x = 2aq$ and $y = aq^2$ into the equation of PQ:
$2aq + paq^2 - 2ap - ap^3 = 0$,
$a\left[2(q-p) + p\left(q^2 - p^2\right)\right] = 0$,
$a\left[2(q-p) + p(q-p)(q+p)\right] = 0$,
$a(q-p)\left[2 + p(q+p)\right] = 0$.
Since $q - p \ne 0$ $\because q \ne p$, and $a \ne 0$,
$\therefore 2 + p(q+p) = 0$, hence $p^2 + pq + 2 = 0$.

Q5dii Gradient of $OP = \frac{ap^2}{2ap} = \frac{p}{2}$,

gradient of $OQ = \frac{aq^2}{2aq} = \frac{q}{2}$.

If $OP \perp OQ$, then $\frac{p}{2} \times \frac{q}{2} = -1$, $\therefore pq = -4$.
From Q5di, $p^2 + pq + 2 = 0$, $\therefore p^2 - 4 + 2 = 0$, hence $p^2 = 2$.

Q6ai $x = \sqrt{3}\sin 2t - \cos 2t + 3$, $\dot{x} = 2\sqrt{3}\cos 2t + 2\sin 2t$,
$\ddot{x} = -4\sqrt{3}\sin 2t + 4\cos 2t = -4\left(\sqrt{3}\sin 2t - \cos 2t\right) = -4(x - 3)$.

Q6aii Period $= \frac{2\pi}{2} = \pi$ seconds.

Q6aiii $\dot{x} = 2\sqrt{3}\cos 2t + 2\sin 2t$,
$\dot{x} = A\cos(2t - \alpha) = A\cos 2t\cos\alpha + A\sin 2t\sin\alpha$.
$\therefore A\cos\alpha = 2\sqrt{3}$ and $A\sin\alpha = 2$.
Hence $A^2\cos^2\alpha + A^2\sin^2\alpha = 16$,
$\therefore A = 4$ and $\sin\alpha = \frac{1}{2}$, i.e. $\alpha = \frac{\pi}{6}$.

$\therefore \dot{x} = 4\cos\left(2t - \frac{\pi}{6}\right)$.

Q6aiv $0 \le t \le \pi$, $0 \le 2t \le 2\pi$.
When $\dot{x} = 2$, $4\cos\left(2t - \frac{\pi}{6}\right) = 2$, $\cos\left(2t - \frac{\pi}{6}\right) = \frac{1}{2}$,

$2t - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$. $\therefore t = \frac{\pi}{4}, \frac{11\pi}{12}$.

When $\dot{x} = -2$, $4\cos\left(2t - \frac{\pi}{6}\right) = -2$, $\cos\left(2t - \frac{\pi}{6}\right) = -\frac{1}{2}$,

$2t - \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}$. $\therefore t = \frac{5\pi}{12}, \frac{3\pi}{4}$.

Q6bi $f(x)=e^{x}-e^{-x}$, $f'(x)=e^{x}+e^{-x}>0$ for all values of x, $\therefore f(x)$ is increasing for all values of x.

Q6bii Equation of $f(x)$: $y=e^{x}-e^{-x}$

Equation of inverse: $x=e^{y}-e^{-y}$.

$xe^{y}=(e^{y})^{2}-1$, $(e^{y})^{2}-xe^{y}-1=0$.

Apply quadratic formula: $e^{y}=\dfrac{x+\sqrt{x^{2}+4}}{2}$ since $e^{y}>0$.

$\therefore y=\log_{e}\left(\dfrac{x+\sqrt{x^{2}+4}}{2}\right)$, $\therefore f^{-1}(x)=\log_{e}\left(\dfrac{x+\sqrt{x^{2}+4}}{2}\right)$.

Q6biii $y=e^{x}-e^{-x}$, $\therefore x=\log_{e}\left(\dfrac{y+\sqrt{y^{2}+4}}{2}\right)$.

When $y=5$, $x=\log_{e}\left(\dfrac{5+\sqrt{5^{2}+4}}{2}\right)=1.65$.

Q7ai $y=kx^{n}$, $\dfrac{dy}{dx}=nkx^{n-1}$. $y=\log_{e}x$, $\dfrac{dy}{dx}=\dfrac{1}{x}$.

At $x=a$, $nka^{n-1}=\dfrac{1}{a}$, $\therefore a^{n}=\dfrac{1}{nk}$.

Q7aii At $x=a$, $ka^{n}=\log_{e}a$.

From Q7ai, $a^{n}=\dfrac{1}{nk}$, $\therefore k\left(\dfrac{1}{nk}\right)=\log_{e}a$, $\log_{e}a=\dfrac{1}{n}$, $a=e^{\frac{1}{n}}$.

$\therefore a^{n}=e=\dfrac{1}{nk}$. Hence $k=\dfrac{1}{en}$.

Q7bi $x=14t\cos\theta$, $\therefore t=\dfrac{x}{14\cos\theta}$.

$y=14t\sin\theta-4.9t^{2}$, $y=14\left(\dfrac{x}{14\cos\theta}\right)\sin\theta-4.9\left(\dfrac{x}{14\cos\theta}\right)^{2}$

$=x\tan\theta-\dfrac{x^{2}}{40\cos^{2}\theta}=x\tan\theta-\dfrac{x^{2}}{40}\sec^{2}\theta$

$=x\tan\theta-\dfrac{x^{2}}{40}(1+\tan^{2}\theta)=mx-\left(\dfrac{1+m^{2}}{40}\right)x^{2}$, where $m=\tan\theta$.

Q7bii At $x=10$ and when $m=2\pm\sqrt{3-0.4h}$,

$y=mx-\left(\dfrac{1+m^{2}}{40}\right)x^{2}$

$=10\left(2\pm\sqrt{3-0.4h}\right)-2.5\left(1+\left(2\pm\sqrt{3-0.4h}\right)^{2}\right)=h$.

Since $3-0.4h\geq 0$, $h\leq\dfrac{3}{0.4}=7.5$, $\therefore$ max h is 7.5 m.

Q7biii Given $m=2\pm\sqrt{3-0.4h}$.

When $h=3.9$, $m=0.8$ or 3.2.

When $h=5.9$, $m=1.2$ or 2.8.

$\therefore$ The other interval is $0.8\leq m\leq 1.2$.

Q7biv Let $y=0$ to find the range.

$mx-\left(\dfrac{1+m^{2}}{40}\right)x^{2}=0$, $x\left[m-\left(\dfrac{1+m^{2}}{40}\right)x\right]=0$,

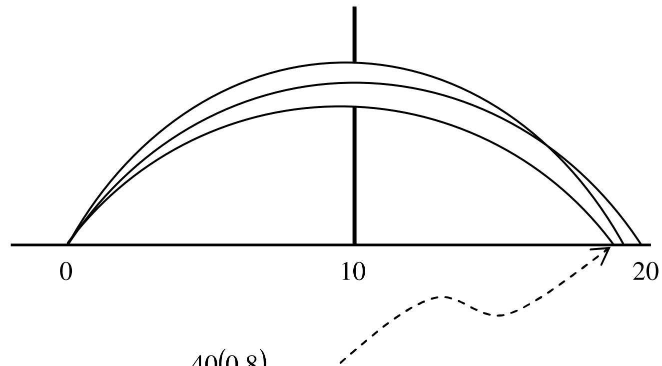
$\therefore x=0$ or $x=\dfrac{40m}{1+m^{2}}$. $\therefore$ the range is $\dfrac{40m}{1+m^{2}}$ metres.

For $2.8\leq m\leq 3.2$, $\dfrac{40(3.2)}{1+3.2^{2}}\leq x\leq\dfrac{40(2.8)}{1+2.8^{2}}$,

$11.388\leq x\leq 12.670$.

Width of interval $=12.670-11.388\approx 1.282$ metres.

The range is maximum when $\theta^{\circ}=45^{\circ}$, i.e. $m=\tan 45^{\circ}=1$, which is within the interval $0.8\leq m\leq 1.2$.



When $m=0.8$, $x=\dfrac{40(0.8)}{1+0.8^{2}}=19.512$; when $m=1$, $x=20$;

when $m=1.2$, $x=19.672$.

Width of interval $=20-19.512\approx 0.488$ metres.

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