NSW BOS Mathematics Extension 2 Solutions 2006
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Q1a Let $u = 9 - 4x^2$, $\frac{du}{dx} = -8x$, $x = -\frac{1}{8}\frac{du}{dx}$.

$\int \frac{x}{\sqrt{9-4x^2}}dx = \int -\frac{1}{8}\frac{1}{\sqrt{u}}\frac{du}{dx}dx = \int -\frac{1}{8}\frac{1}{\sqrt{u}}du$

$= -\frac{1}{4}\sqrt{u} + c = -\frac{1}{4}\sqrt{9-4x^2} + c$.

Q1b $\int \frac{dx}{x^2 - 6x + 13} = \frac{1}{2}\int \frac{2}{4 + (x-3)^2}dx = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + c$.

Q1ci $\frac{16x - 43}{(x-3)^2(x+2)} = \frac{a}{(x-3)^2} + \frac{b}{x-3} + \frac{c}{x+2}$

$= \frac{a(x+2) + b(x-3)(x+2) + c(x-3)^2}{(x-3)^2(x+2)}$.

$\therefore 16x - 43 = a(x+2) + b(x-3)(x+2) + c(x-3)^2$ for all $x \in R \setminus \{-2, 3\}$

$\lim_{x\to -2}(16x - 43) = \lim_{x\to -2} a(x+2) + b(x-3)(x+2) + c(x-3)^2$, $c = -3$

$\lim_{x\to 3}(16x - 43) = \lim_{x\to 3} a(x+2) + b(x-3)(x+2) + c(x-3)^2$, $a = 1$.

When $x = 0$, $b = 3$.

Q1cii $\int \frac{16x-43}{(x-3)^2(x+2)}dx = \int\left(\frac{1}{(x-3)^2} + \frac{3}{x-3} - \frac{3}{x+2}\right)dx$

$= -\frac{1}{x-3} + 3\log_e|x-3| - 3\log_e|x+2| + c$

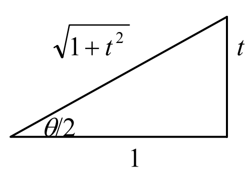
$= -\frac{1}{x-3} + 3\log_e\left|\frac{x-3}{x+2}\right| + c$.

Q1d Let $u = t$, then $\frac{du}{dt} = 1$. Let $\frac{dv}{dt} = e^{-t}$, then $v = -e^{-t}$.

$\int_0^2 te^{-t}dt = [uv]_0^2 - \int_0^2 v\frac{du}{dt}dt = \left[-te^{-t}\right]_0^2 + \int_0^2 e^{-t}dt$

$= -2e^{-2} + \left[-e^{-t}\right]_0^2 = -2e^{-2} - e^{-2} + 1 = 1 - 3e^{-2}$.

Q1e Let $t = \tan\frac{\theta}{2}$,



$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right) = \frac{2t}{1+t^2}$.

$\theta = 2\tan^{-1}t$, $\frac{d\theta}{dt} = \frac{2}{1+t^2}$, $\frac{dt}{d\theta} = \frac{1+t^2}{2}$.

$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{\sin\theta}d\theta = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1+t^2}{2t}d\theta = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{t}\frac{dt}{d\theta}d\theta = \int_1^{\sqrt{3}}\frac{1}{t}dt$

$= \left[\log_e|t|\right]_1^{\sqrt{3}} = \log_e\sqrt{3} = \frac{1}{2}\log_e 3$.

Q2a Given $z = 3 + i$, $w = 2 - 5i$.

Q2ai $z^2 = (3+i)^2 = 8 + 6i$

Q2aii $\bar{z}w = (3-i)(2-5i) = 1 - 17i$

Q2aiii $\frac{w}{z} = \frac{w\bar{z}}{z\bar{z}} = \frac{(2-5i)(3-i)}{(3+i)(3-i)} = \frac{1-17i}{10} = 0.1 - 1.7i$

Q2bi $\left|\sqrt{3} - i\right| = \sqrt{3+1} = 2$, $Arg\left(\sqrt{3} - i\right) = Tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$,

$\therefore \sqrt{3} - i = 2cis\left(-\frac{\pi}{6}\right)$.

Q2bii

$\therefore \left(\sqrt{3} - i\right)^7 = \left(2cis\left(-\frac{\pi}{6}\right)\right)^7 = 128cis\left(-\frac{7\pi}{6}\right) = 128cis\left(\frac{5\pi}{6}\right)$.

Q2biii $\therefore \left(\sqrt{3} - i\right)^7 = 128cis\left(\frac{5\pi}{6}\right) = 128\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 128\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -64\sqrt{3} + 64i$

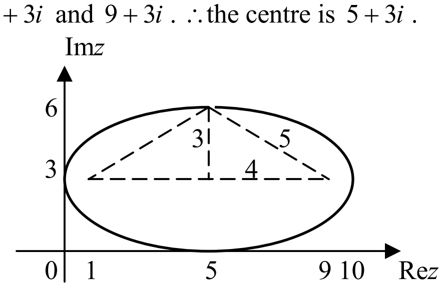
Q2c Let $z^3 = -1 = 1cis(\pi + 2k\pi)$, $\therefore z = \sqrt[3]{-1} = 1cis\left(\frac{\pi + 2k\pi}{3}\right)$.

When $k = -1$, $z = 1cis\left(-\frac{\pi}{3}\right)$.

When $k = 0$, $z = 1cis\left(\frac{\pi}{3}\right)$.

When $k = 1$, $z = 1cis(\pi)$.

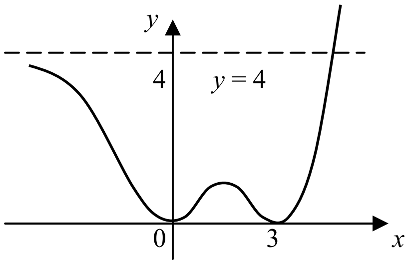
Q2di Foci are $1 + 3i$ and $9 + 3i$. $\therefore$ the centre is $5 + 3i$.



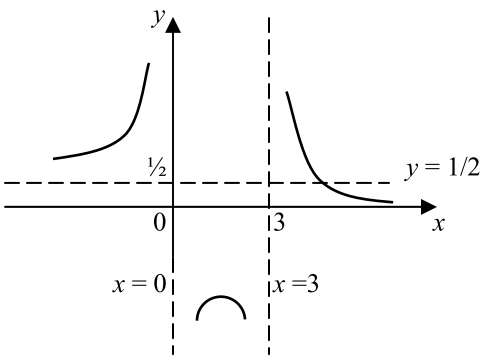
Q2dii Length of minor axis $= 6$; length of major axis $= 10$.

Q2diii $\left[0, \frac{\pi}{2}\right]$

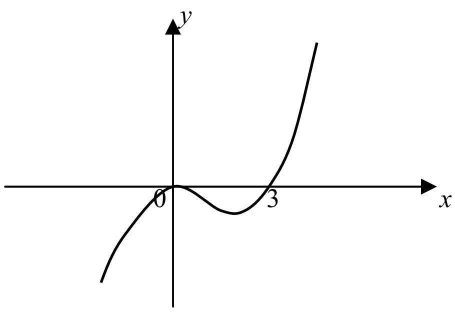
Q3ai $y = (f(x))^2$



Q3aii $y = \frac{1}{f(x)}$



Q3aiii $y = xf(x)$



Q3bi Let $y = 0$, $x = \pm 12$, x-intercepts are $\therefore (-12,0)$, $(12,0)$.

Q3bii $\pm ae = \pm\sqrt{a^2 + b^2} = \pm\sqrt{144 + 25} = \pm 13$, Foci are $\therefore (-13,0)$, $(13,0)$.

Q3biii Directrices: $x = \pm\frac{a}{e} = \pm\frac{a^2}{ae}$, $\therefore x = \pm\frac{144}{13}$.

Asymptotes: $y = \pm\frac{b}{a}x$, $\therefore y = \pm\frac{5}{12}x$.

Q3ci The zeros are $a + bi$, $a - bi$, $a + 2bi$ and $a - 2bi$.
Sum of zeros $= 4a = 12$, $\therefore a = 3$
Product of zeros $= (a^2 + b^2)(a^2 + 4b^2) = 130$,
$(9 + b^2)(9 + 4b^2) = 130$, $\therefore 4(b^2)^2 + 45b^2 - 49 = 0$.
Hence $(4b^2 + 49)(b^2 - 1) = 0$. Since b is real and $b > 0$,
$\therefore b = 1$.

Q3cii $P(x) = (x - 3 - i)(x - 3 + i)(x - 3 - 2i)(x - 3 + 2i)$
$= (x^2 - 6x + 10)(x^2 - 6x + 13)$

Q4a $p(x) = ax^3 + bx + c$, $p'(x) = 3ax^2 + b$.
$p(1) = a + b + c = 0$
$p(-1) = -a - b + c = 4$
$p'(1) = 3a + b = 0$.
Solve for a, b and c. $a = 1$, $b = -3$ and $c = 2$.

Q4b Area of vertical cross-section $= (2x)(2x) = 4x^2 = 4y$.

Volume of solid $= \int_0^1 4y\,dy = \left[2y^2\right]_0^1 = 2$ cubic units.

Q4ci $m_{PQ} = \frac{\frac{1}{q} - \frac{1}{p}}{q - p} = -\frac{1}{pq}$, $\therefore m_{\perp PQ} = pq$. Point $R\left(r, \frac{1}{r}\right)$.

Equation of line ℓ: $y - \frac{1}{r} = pq(x - r)$,

i.e. $y = pqx - pqr + \frac{1}{r}$ …… (1)

Q4cii Similarly, equation of line m: $y = qrx - pqr + \frac{1}{p}$ …(2)

Q4ciii Solve the equations of lines ℓ and m simultaneously to find T.

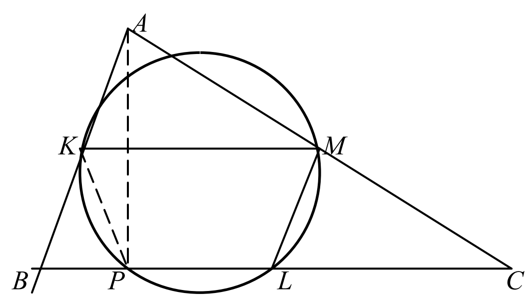
(1) − (2): $pqx - qrx + \frac{1}{r} - \frac{1}{p} = 0$, $\therefore x = -\frac{1}{pqr}$ …… (3)

Substitute (3) in (1): $y = -pqr$.

$\therefore T$ is $\left(-\frac{1}{pqr}, -pqr\right)$ and it satisfies the equation $xy = 1$.

Hence T lies on the hyperbola $xy = 1$.

Q4di

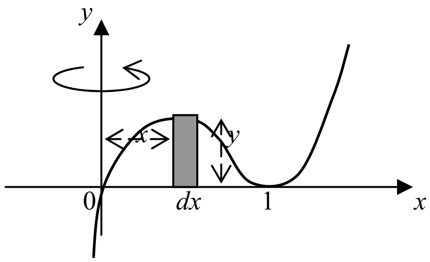


Since K and M are midpoints, $\therefore KM \parallel BL$.
Since L and M are midpoints, $\therefore LM \parallel BK$.
$\therefore KMLB$ is a parallelogram.

Q4dii $\angle KPB + \angle KPL = \angle KML + \angle KPL = 180^0$,
$\therefore \angle KPB = \angle KML$.

Q4diii $KMLB$ is a parallelogram, $\therefore \angle KBP = \angle KML$,
$\therefore \angle KPB = \angle KBP$. $\therefore KP = KB = KA$, $\therefore \angle KPA = \angle KAP$.
Consider ΔAPB, $\angle KPA + \angle KPB + \angle KAP + \angle KBP = 180^0$.
$\therefore \angle KPA + \angle KPB + \angle KPA + \angle KPB = 180^0$,
$\therefore 2(\angle KPA + \angle KPB) = 180^0$, $\therefore \angle KPA + \angle KPB = 90^0$.
Hence $AP \perp BC$.

Q5a



Volume of cylindrical shell $= 2\pi xydx$.

Volume of solid $= \int_0^1 2\pi xydx = \int_0^1 2\pi x^2(x-1)^2\,dx$

$= 2\pi\int_0^1 \left(x^4 - 2x^3 + x^2\right)dx = 2\pi\left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{15}$ cubic units.

Q5bi $\cos(\alpha+\beta) + \cos(\alpha-\beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= 2\cos\alpha\cos\beta$

Q5bii $(\cos 4\theta + \cos 2\theta) + (\cos 3\theta + \cos\theta) = 0$, $0 \le \theta \le 2\pi$.

$2\cos 3\theta\cos\theta + 2\cos 2\theta\cos\theta = 0$, $2\cos\theta(\cos 3\theta + \cos 2\theta) = 0$,

$2\cos\theta\left(2\cos\frac{5\theta}{2}\cos\frac{\theta}{2}\right) = 0$, $\therefore 4\cos\theta\cos\frac{5\theta}{2}\cos\frac{\theta}{2} = 0$.

Hence $\cos\theta = 0$, $\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}$ or

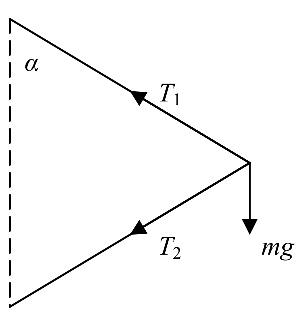
$\cos\frac{5\theta}{2} = 0$, $\frac{5\theta}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$,

$\therefore \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$ or

$\cos\frac{\theta}{2} = 0$, $\frac{\theta}{2} = \frac{\pi}{2}$, $\therefore \theta = \pi$.

Solution set is $\left\{\frac{\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{3\pi}{2}, \frac{9\pi}{5}\right\}$.

Q5ci



Vertical component: $T_1\cos\alpha - T_2\cos\alpha - mg = 0$,

Horizontal component: $T_1\sin\alpha + T_2\sin\alpha = m\omega^2\ell\sin\alpha$,

$\therefore T_1 + T_2 = m\omega^2\ell$.

Q5cii If $T_2 = 0$, $T_1\cos\alpha = mg$ and $T_1 = m\omega^2\ell$,

$\therefore \omega^2 = \frac{g}{\ell\cos\alpha}$, $\therefore \omega = \sqrt{\frac{g}{\ell\cos\alpha}}$.

Q5di $3\times3\times3\times3 = 81$

Q5dii $\Pr(WDLD) = 0.2\times0.6\times0.2\times0.6 = 0.0144$

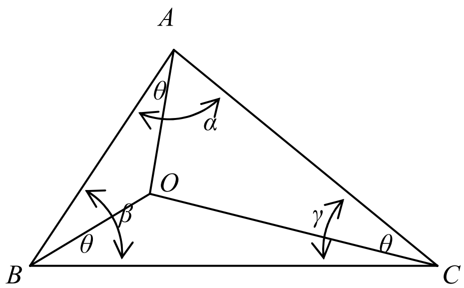
Q5diii $\Pr(more\_po\text{int}\,s) = \Pr(4W) + \Pr(3W1D) + \Pr(3W1L)$

$+ \Pr(2W2D) + \Pr(2W1D1L) + \Pr(1W3D)$

$= 0.2^4 + \frac{4!}{3!}(0.2^3)(0.6) + \frac{4!}{3!}(0.2^3)(0.2) + \frac{4!}{2!2!}(0.2)^2(0.6)^2$

$+ \frac{4!}{2!}(0.2)^2(0.6)(0.2) + \frac{4!}{3!}(0.2)(0.6^3) = 0.344$.

Q6



Q6ai Consider ΔOAB, the sine rule: $\frac{OA}{\sin(\beta-\theta)} = \frac{OB}{\sin\theta}$,

$\therefore \frac{OA}{OB} = \frac{\sin(\beta-\theta)}{\sin\theta}$.

Q6aii Similarly for ΔOBC and ΔOCA,

$\frac{OB}{OC} = \frac{\sin(\gamma-\theta)}{\sin\theta}$ and $\frac{OC}{OA} = \frac{\sin(\alpha-\theta)}{\sin\theta}$.

$\therefore \frac{OA}{OB}\times\frac{OB}{OC}\times\frac{OC}{OA} = \frac{\sin(\beta-\theta)}{\sin\theta}\times\frac{\sin(\gamma-\theta)}{\sin\theta}\times\frac{\sin(\alpha-\theta)}{\sin\theta}$

$\therefore 1 = \frac{\sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta)}{\sin^3\theta}$

and hence $\sin^3\theta = \sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta)$.

Q6aiii $\cot x - \cot y = \frac{\cos x}{\sin x} - \frac{\cos y}{\sin y} = \frac{\cos x\sin y - \cos y\sin x}{\sin x\sin y}$

$= \frac{\sin(y-x)}{\sin x\sin y}$.

Q6aiv $(\cot\theta - \cot\alpha)(\cot\theta - \cot\beta)(\cot\theta - \cot\gamma)$

$= \frac{\sin(\alpha-\theta)}{\sin\theta\sin\alpha}\times\frac{\sin(\beta-\theta)}{\sin\theta\sin\beta}\times\frac{\sin(\gamma-\theta)}{\sin\theta\sin\gamma}$

$= \frac{\sin^3\theta}{\sin^3\theta\sin\alpha\sin\beta\sin\gamma} = \frac{1}{\sin\alpha\sin\beta\sin\gamma}$

$= \cos ec\alpha\cos ec\beta\cos ec\gamma$

Q6av Without loss of generality, let $\alpha = \beta = 45^0$ and $\gamma = 90^0$.

$(\cot\theta - \cot 45^0)(\cot\theta - \cot 45^0)(\cot\theta - \cot 90^0)$

$= \cos ec45^0\cos ec45^0\cos ec90^0$,

$\therefore \cot\theta(\cot\theta - 1)^2 = 2$, $\cot^3\theta - 2\cot^2\theta + \cot\theta - 2 = 0$,

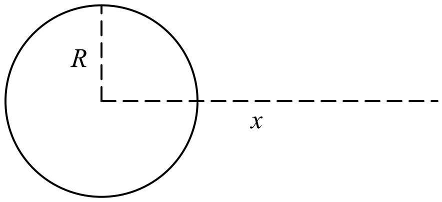
$(\cot^3\theta - 2\cot^2\theta) + (\cot\theta - 2) = 0$,

$\cot^2\theta(\cot\theta - 2) + (\cot\theta - 2) = 0$, $(\cot\theta - 2)(\cot^2\theta + 1) = 0$.

Since $\cot^2\theta + 1 \ne 0$, $\therefore \cot\theta - 2 = 0$, $\cot\theta = 2$, $\tan\theta = 0.5$,

$\theta = \tan^{-1}(0.5)$.

Q6b



Given at $t=0$, $x=R$ and $v=+u$ where u is a positive real value.

Equation of motion: $\ddot{x}=-\frac{k}{x^3}$, $x\geq R$.

Q6bi At $x=R$, $\ddot{x}=-g$, $\therefore -\frac{k}{R^3}=-g$, $k=gR^3$.

Q6bii $\ddot{x}=-\frac{k}{x^3}$, $v\frac{dv}{dx}=-\frac{k}{x^3}$,

$\int_R^x v\frac{dv}{dx}dx=\int_R^x\left(-\frac{k}{x^3}\right)dx$, $\int_u^v vdv=\int_R^x\left(-\frac{k}{x^3}\right)dx$,

$\left[\frac{v^2}{2}\right]_u^v=\left[\frac{k}{2x^2}\right]_R^x$, $\frac{v^2}{2}-\frac{u^2}{2}=\frac{k}{2x^2}-\frac{k}{2R^2}$,

$v^2-u^2=\frac{gR^3}{x^2}-gR$, $\therefore v^2=\frac{gR^3}{x^2}-\left(gR-u^2\right)$.

Q6biii $x=\sqrt{R^2+2uRt-\left(gR-u^2\right)t^2}$.

When $u=gR$, $x=\sqrt{R^2+2uRt}$.

As $t\to\infty$, $x\to\infty$.

$x=\sqrt{R^2+2uRt-\left(gR-u^2\right)t^2}=t\sqrt{\frac{R^2}{t^2}+\frac{2uR}{t}-gR+u^2}$

As $t\to\infty$, $x\to t\sqrt{u^2-gR}$, which is real and ∞ if $u^2>gR$, i.e. $u>\sqrt{gR}$.

Hence for $u\geq\sqrt{gR}$, the particle will not return to the planet.

Q6biv1 $v^2=\frac{gR^3}{x^2}-\left(gR-u^2\right)$.

If $u<\sqrt{gR}$, $v=0$ at $x=D$, $\therefore 0=\frac{gR^3}{D^2}-\left(gR-u^2\right)$,

$\therefore D=\sqrt{\frac{gR^3}{gR-u^2}}=R\sqrt{\frac{gR}{gR-u^2}}$.

Q6biv2

$x=\sqrt{R^2+2uRt-\left(gR-u^2\right)t^2}$, $x^2=R^2+2uRt-\left(gR-u^2\right)t^2$.

At $x=R$, $R^2=R^2+2uRt-\left(gR-u^2\right)t^2$,

$\therefore 2uRt-\left(gR-u^2\right)t^2=0$, $\therefore t\left(2uR-\left(gR-u^2\right)t\right)=0$.

$\therefore t=0$, i.e. the time when the particle is projected upward, or $t=\frac{2uR}{gR-u^2}$, i.e. the time when the particle returns to the surface. The time interval for the complete journey is $\frac{2uR}{gR-u^2}$.

The time for one-way journey in either direction is $\frac{uR}{gR-u^2}$.

Q7ai $y=\cos x$ and $y=\tan x$ intersect at $P(\alpha,...)$,

$\therefore \cos\alpha=\tan\alpha$.

$m_1=\frac{dy}{dx}=-\sin x$, $m_2=\frac{dy}{dx}=\sec^2 x$.

At P, $m_1m_2=-\sin\alpha\sec^2\alpha=-\frac{\sin\alpha}{\cos^2\alpha}=-\frac{\tan\alpha}{\cos\alpha}=-1$,

$\therefore$ the curves intersect at right angle at P.

Q7aii $\cos\alpha=\tan\alpha$, $\cos^2\alpha=\tan^2\alpha=\sec^2\alpha-1$,

$\therefore \frac{1}{\sec^2\alpha}=\sec^2\alpha-1$, $\therefore \left(\sec^2\alpha\right)^2-\sec^2\alpha-1=0$,

$\therefore \sec^2\alpha=\frac{1+\sqrt{1+4}}{2}=\frac{1+\sqrt{5}}{2}$. Note: $\frac{1-\sqrt{5}}{2}$ is rejected because $\sec^2\alpha>0$.

Q7bi $I_n=\int_0^x\sec^n t\,dt$, where $0\leq x\leq\frac{\pi}{2}$.

Let $u=\sec^{n-2}t$ and $\frac{dv}{dt}=\sec^2 t$. $\therefore \frac{du}{dt}=(n-2)\sec^{n-2}t\tan t$ and $v=\tan t$.

$I_n=\int_0^x\sec^n t\,dt=\int_0^x\sec^{n-2}t\sec^2 t\,dt=\int_0^x u\frac{dv}{dt}dt$

$=[uv]_0^x-\int_0^x v\frac{du}{dt}dt=\left[\sec^{n-2}t\tan t\right]_0^x-(n-2)\int_0^x\sec^{n-2}t\tan^2 t\,dt$

$=\sec^{n-2}x\tan x-(n-2)\int_0^x\sec^{n-2}t\left(\sec^2 t-1\right)dt$

$=\sec^{n-2}x\tan x-(n-2)\int_0^x\sec^n t\,dt+(n-2)\int_0^x\sec^{n-2}t\,dt$.

$\therefore I_n=\sec^{n-2}x\tan x-(n-2)I_n+(n-2)I_{n-2}$,

$\therefore (n-1)I_n=\sec^{n-2}x\tan x+(n-2)I_{n-2}$,

$\therefore I_n=\frac{\sec^{n-2}x\tan x}{n-1}+\frac{n-2}{n-1}I_{n-2}$.

Q7bii $\int_0^{\frac{\pi}{3}}\sec^4 t\,dt=I_4=\frac{\sec^2\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)}{3}+\frac{2}{3}I_2$

$=\frac{\sec^2\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)}{3}+\frac{2}{3}\tan\left(\frac{\pi}{3}\right)$

$=\frac{4\sqrt{3}}{3}+\frac{2\sqrt{3}}{3}=2\sqrt{3}$.

Q7ci Given $x_1=1$ and $x_{n+1}=\frac{4+x_n}{1+x_n}$ for $n\geq1$.

Prove that for $n\geq1$, $x_n=2\left(\frac{1+\alpha^n}{1-\alpha^n}\right)$, where $\alpha=-\frac{1}{3}$.

It is true for $n=1$, $x_1=2\left(\frac{1-\frac{1}{3}}{1+\frac{1}{3}}\right)=1$.

Assume that it is true for $n = k$, i.e. $x_k = 2\left(\frac{1+\alpha^k}{1-\alpha^k}\right)$,

$$x_{k+1} = \frac{4+x_k}{1+x_k} = \frac{4+2\left(\frac{1+\alpha^k}{1-\alpha^k}\right)}{1+2\left(\frac{1+\alpha^k}{1-\alpha^k}\right)} = \frac{4\left(1-\alpha^k\right)+2\left(1+\alpha^k\right)}{1-\alpha^k+2\left(1+\alpha^k\right)}$$

$$= \frac{6-2\alpha^k}{3+\alpha^k} = 2\left(\frac{3-\alpha^k}{3+\alpha^k}\right) = 2\left(\frac{1-\frac{1}{3}\alpha^k}{1+\frac{1}{3}\alpha^k}\right) = 2\left(\frac{1+\alpha^{k+1}}{1-\alpha^{k+1}}\right).$$

It is also true for $n = k+1$.

$\therefore$ it is true for all $n \ge 1$.

Q7cii As $n \to \infty$, $\alpha^n = \left(-\frac{1}{3}\right)^n \to 0$, $x_n \to 2$.

Q8ai Given $0 \le t \le \frac{1}{\sqrt{2}}$, then $0 \le t^2 \le \frac{1}{2}$, $\therefore \frac{1}{2} \le 1-t^2 \le 1$,

$\therefore 1 \le \frac{1}{1-t^2} \le 2$, hence $0 \le \frac{1}{1-t^2} \le 2$.

Multiply by $2t^2$, $0 \le \frac{2t^2}{1-t^2} \le 4t^2$.

Q8aii $0 \le \frac{2t^2}{1-t^2} \le 4t^2$, $0 \le \frac{2-2\left(1-t^2\right)}{1-t^2} \le 4t^2$,

$0 \le \frac{2}{1-t^2} - 2 \le 4t^2$, $0 \le \frac{2}{(1+t)(1-t)} - 2 \le 4t^2$,

$0 \le \frac{1}{1+t} + \frac{1}{1-t} - 2 \le 4t^2$.

Q8aiii $0 \le \int_0^x \left(\frac{1}{1+t} + \frac{1}{1-t} - 2\right) dt \le \int_0^x 4t^2 \, dt$,

$0 \le \left[\log_e\left|\frac{1+t}{1-t}\right| - 2t\right]_0^x \le \left[\frac{4t^3}{3}\right]_0^x$, where $0 \le x \le \frac{1}{\sqrt{2}}$,

$0 \le \log_e\left(\frac{1+x}{1-x}\right) - 2x \le \frac{4x^3}{3}$,

Q8aiv $\therefore e^0 \le e^{\log_e\left(\frac{1+x}{1-x}\right)-2x} \le e^{\frac{4x^3}{3}}$, $\therefore 1 \le e^{\log_e\left(\frac{1+x}{1-x}\right)} e^{-2x} \le e^{\frac{4x^3}{3}}$,

$\therefore 1 \le \left(\frac{1+x}{1-x}\right) e^{-2x} \le e^{\frac{4x^3}{3}}$.

Q8bi $f(x) = x^n e^{-x}$, $n \ge 2$.

$f'(x) = nx^{n-1}e^{-x} - x^n e^{-x}$,

$f''(x) = n(n-1)x^{n-2}e^{-x} - nx^{n-1}e^{-x} - nx^{n-1}e^{-x} + x^n e^{-x}$

$f''(x) = x^{n-2}e^{-x}\left(n(n-1) - 2nx + x^2\right)$.

At the inflexion points $x = a$ and $x = b$, $f''(x) = 0$.

Since $x^{n-2} > 0$ and $e^{-x} > 0$, $\therefore n(n-1) - 2nx + x^2 = 0$.

Use the quadratic formula to solve for *x* in terms of *n*:

$x = n \pm \sqrt{n}$.

Hence $a = n - \sqrt{n}$ and $b = n + \sqrt{n}$.

Q8bii $\frac{f(b)}{f(a)} = \frac{\left(n+\sqrt{n}\right)^n e^{-\left(n+\sqrt{n}\right)}}{\left(n-\sqrt{n}\right)^n e^{-\left(n-\sqrt{n}\right)}} = \left(\frac{n+\sqrt{n}}{n-\sqrt{n}}\right)^n e^{-2\sqrt{n}}$

$$= \left(\frac{1+\frac{\sqrt{n}}{n}}{1-\frac{\sqrt{n}}{n}}\right)^n e^{-2\sqrt{n}} = \left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right)^n e^{-2\sqrt{n}}$$

Q8biii From Q8aiv, $1 \le \left(\frac{1+x}{1-x}\right)e^{-2x} \le e^{\frac{4x^3}{3}}$.

Let $x = \frac{1}{\sqrt{n}}$, $\therefore 1 \le \left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right) e^{-\frac{2}{\sqrt{n}}} \le e^{\frac{4}{3n\sqrt{n}}}$,

$$\therefore 1^n \le \left(\left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right) e^{-\frac{2}{\sqrt{n}}}\right)^n \le \left(e^{\frac{4}{3n\sqrt{n}}}\right)^n,$$

$$\therefore 1 \le \left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right)^n e^{-2\sqrt{n}} \le e^{\frac{4}{3\sqrt{n}}},$$

$\therefore 1 \le \frac{f(b)}{f(a)} \le e^{\frac{4}{3\sqrt{n}}}$.

Q8biv $\frac{f(b)}{f(a)}$ is sandwiched between 1 and $e^{\frac{4}{3\sqrt{n}}}$.

As $n \to \infty$, $e^{\frac{4}{3\sqrt{n}}} \to e^0 = 1$, $\therefore \frac{f(b)}{f(a)} \to 1$.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors.