NSW BOS Mathematics Solutions 2006
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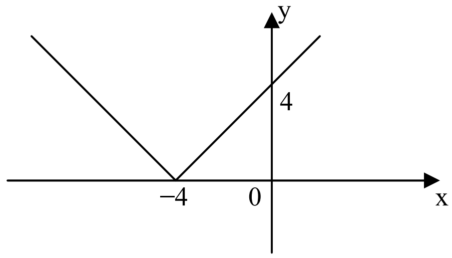
Q1a $e^{-0.5} = 0.607$

Q1b $2x^2 + 5x - 3 = (2x-1)(x+3)$

Q1c



Q1d $\frac{\sin\theta}{5} = \frac{\sin 33^0}{9}$, $\theta = 18^0$

Q1e $3 - 5x \le 2$, $-5x \le -1$, $x \ge \frac{-1}{-5}$, $x \ge \frac{1}{5}$

Q1f $a = \frac{13}{5}$, $r = \frac{1}{5}$, $S_\infty = \frac{\frac{13}{5}}{1-\frac{1}{5}} = \frac{13}{4}$.

Q2ai Let $y = x\tan x$, $\frac{dy}{dx} = \tan x + x\sec^2 x$

Q2aii Let $y = \frac{\sin x}{x+1} = \frac{(x+1)\cos x - \sin x}{(x+1)^2}$

Q2bi $\int\left(1+e^{7x}\right)dx = x + \frac{1}{7}e^{7x} + c$

Q2bii Let $u = 1 + x^2$, $\frac{du}{dx} = 2x$. When $x = 0$, $u = 1$. When $x = 3$, $u = 10$.

$$\int_0^3 \frac{8x}{1+x^2}dx = \int_0^3 \frac{4}{u}\frac{du}{dx}dx = \int_1^{10}\frac{4}{u}du = \left[4\log_e u\right]_1^{10} = 4\log_e 10 .$$

Q2c $y = \cos 2x$, $\frac{dy}{dx} = -2\sin 2x$.

When $x = \frac{\pi}{6}$, $y = \frac{1}{2}$, $\frac{dy}{dx} = -\sqrt{3}$.

Equation of tangent: $y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\pi}{6}\right)$,

$y = -\sqrt{3}x + \frac{\pi\sqrt{3}}{6} + \frac{1}{2}$.

Q3ai $m = \frac{-4-4}{5-1} = -2$, $\therefore y = -2x + c$.

Use $(1,4)$ to find c: $4 = -2(1) + c$, $c = 6$, $\therefore y = -2x + 6$.

Q3aii $D(0,6)$

Q3aiii Let $P(a,b)$ be a point on line AB such that $CP \perp AB$.

$\therefore 2a + b - 6 = 0$, and $\frac{b-(-1)}{a-(-3)} \times (-2) = -1$. Solve the two equations to find $a = \frac{11}{5}$ and $b = \frac{8}{5}$.

Perpendicular distance

$$CP = \sqrt{\left(\frac{11}{5}-(-3)\right)^2 + \left(\frac{8}{5}-(-1)\right)^2} = \frac{13\sqrt{5}}{5} .$$

Q3aiv $AD = \sqrt{(1-0)^2 + (4-6)^2} = \sqrt{5}$

Area of $\Delta ADC = \frac{1}{2} \times \sqrt{5} \times \frac{13\sqrt{5}}{5} = \frac{13}{2}$ square units.

Q3b $\sum_{r=2}^{4}\frac{1}{r} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$

Q3ci $a = 560$, $d = -17$, $t_{14} = 560 + (14-1)(-17) = 339$

Q3cii $S_{14} = \frac{14}{2}(560 + 339) = 6293$

Q3ciii $t_n < 60$, $560 + (n-1)(-17) < 60$, $(n-1)(-17) < -500$,

$\therefore n - 1 > \frac{-500}{-17} = 29.4$, $n > 30.4$, i.e. $n = 31$, 31^{st} day.

Q4ai $\angle DAB = \pi - 2\left(\frac{\pi}{6}\right) = \frac{2\pi}{3}$

Q4aii The cosine rule:

$$BD = \sqrt{3^2 + 3^2 - 2(3)(3)\cos\frac{2\pi}{3}} = 3\sqrt{3} \text{ m}$$

Q4aiii Area $ABCD$ = area of triangle ABD + area of sector BCD

$$= \frac{1}{2}(3)(3)\sin\frac{2\pi}{3} + \frac{1}{2}(3)^2\left(\frac{5\pi}{6}\right) = \frac{9}{2}\left(\frac{\sqrt{3}}{2} + \frac{5\pi}{6}\right) \text{ m}^2$$

Q4b $y = x^2 + 1$, $x^2 = y - 1$.

$$V = \int_1^5 \pi x^2 dy = \int_1^5 \pi(y-1)dy = \left[\frac{\pi(y-1)^2}{2}\right]_1^5 = 8\pi \text{ cubic units}$$

Q4ci $\Pr(3w)=\dfrac{\binom{32}{3}}{\binom{64}{3}}=\dfrac{5}{42}$

Q4cii $\Pr(3w\cup 3b)=\Pr(3w)+\Pr(3b)=\dfrac{5}{42}+\dfrac{5}{42}=\dfrac{5}{21}$

Q4ciii $\Pr(notsamecolour)=1-\Pr(3w\cup 3b)=1-\dfrac{5}{21}=\dfrac{16}{21}$

Q5ai $f(x)=2x^2(3-x)=6x^2-2x^3$, $f'(x)=12x-6x^2$.
At T.P., $f'(x)=0$, $12x-6x^2=0$, $6x(2-x)=0$.
$\therefore x=0$ and $y=0$, or $x=2$ and $y=8$.
The turning points are $(0,0)$, $(2,8)$.
Nature:

x	-1	0	1	2	3
$f'(x)$	$-$	0	+	0	$-$

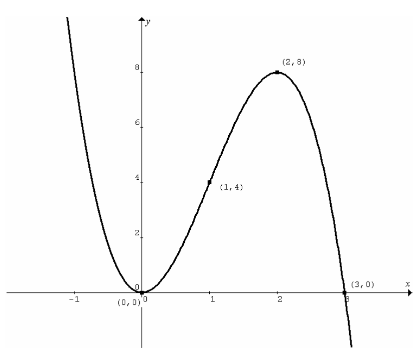
$(0,0)$ local minimum; $(2,8)$ local maximum.

Q5aii $f''(x)=12-12x$. At inflexion point, $f''(x)=0$, $12-12x=0$, $x=1$ and $y=f(1)=4$.

x	0.5	1	1.5
$f'(x)$	+ 4.5	+ 6	+ 4.5

The inflexion point is $(1,4)$.

Q5aiii x-intercept, let $f(x)=0$, $2x^2(3-x)=0$, $x=0$, $x=3$.



Q5aiv Minimum value of $f(x)$ for $-1\le x\le 4$ occurs at $x=4$, minimum value is $f(4)=2(4)^2(3-4)=-32$.

Q5bi $\dfrac{d}{dx}\log_e(\cos x)=\dfrac{1}{\cos x}\times(-\sin x)=-\tan x$.

Q5bii Area $=\displaystyle\int_0^{\frac{\pi}{4}}(\tan x-x)dx=\int_0^{\frac{\pi}{4}}\tan x dx-\int_0^{\frac{\pi}{4}}x dx$

$=\left[-\log_e(\cos x)\right]_0^{\frac{\pi}{4}}-\left[\dfrac{x^2}{2}\right]_0^{\frac{\pi}{4}}=-\log_e\left(\cos\dfrac{\pi}{4}\right)-\dfrac{1}{2}\left(\dfrac{\pi}{4}\right)^2$

$=\log_e\sqrt{2}-\dfrac{\pi^2}{32}$.

Q6ai Given $\angle BAC=\angle CAD$ and $BC\,//\,AD$.
$\angle CAD=\angle BCA$ (Alternate angles), $\therefore \angle BAC=\angle BCA$.

Q6aii Since $\angle BAC=\angle BCA$ and given $\angle ABP=\angle PBC$, $\therefore \angle BPA=\angle BPC$. $\therefore \Delta BPA$ and ΔBPC are similar. Since the corresponding sides are the same side BP, $\therefore \Delta BPA\equiv\Delta BPC$.

Q6aiii Consider ΔBPA and ΔDPC.
Since $\Delta BPA\equiv\Delta BPC$, $\therefore AP=PC$.
Use similar proof as Q6aii, $\Delta BPA\equiv\Delta DPA$, $\therefore BP=PD$.
$\angle BPA=\angle DPC$ (Vertically opposite angles). Since ΔBPA and ΔDPC have two pairs of equal corresponding sides and equal included angles, $\therefore \Delta BPA\equiv\Delta DPC$.
Hence $\Delta BPA\equiv\Delta BPC\equiv\Delta DPA\equiv\Delta DPC$, and the corresponding sides $AB=BC=CD=DA$. $\therefore ABCD$ is a rhombus.

Q6bi $P=150+300e^{-0.05t}$. When $t=0$, $P=150+300=450$.

Q6bii $\dfrac{dP}{dt}=-15e^{-0.05t}$. When $t=10$, $\dfrac{dP}{dt}=-15e^{-0.5}$.

Q6biii As $t\to\infty$, $P\to 150$.

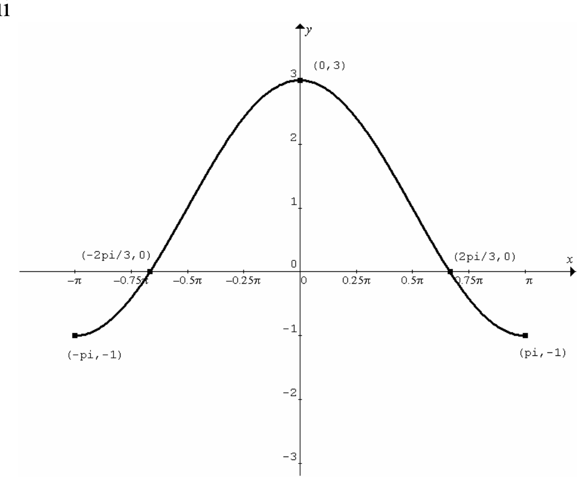
Q6biv $P<200$, $150+300e^{-0.05t}<200$, $300e^{-0.05t}<50$, $e^{-0.05t}<\dfrac{1}{6}$, $-0.05t<\log_e\dfrac{1}{6}$, $t>\dfrac{\log_e\frac{1}{6}}{-0.05}$, $t>35.84$, i.e. 35.84 years after observations began.

Q7ai $x^2-3x+1=(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)+\alpha\beta$.
$\therefore \alpha\beta=1$ and $\alpha+\beta=3$

Q7aii $\alpha\beta=1$, $\therefore \beta=\dfrac{1}{\alpha}$. $\therefore \alpha+\dfrac{1}{\alpha}=\alpha+\beta=3$.

Q7bi $f(x)=1+2\cos x$.
Let $f(x)=0$, $1+2\cos x=0$, $\cos x=-\dfrac{1}{2}$, $x=\dfrac{2\pi}{3}$ is one of the x-intercepts.

Q7bii



Q7biii $\int_{-\frac{\pi}{2}}^{\frac{2\pi}{3}} (1+2\cos x)dx = [x+2\sin x]_{-\frac{\pi}{2}}^{\frac{2\pi}{3}}$

$= \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} - \left(-\frac{\pi}{2}\right) - 2\sin\left(-\frac{\pi}{2}\right) = \frac{2\pi}{3} + \sqrt{3} + \frac{\pi}{2} + 2$

$= \frac{7\pi}{6} + \sqrt{3} + 2$

Q7ci $2x^2 + (k-2)x + 8 = 0$,
$\Delta = (k-2)^2 - 4(2)(8) = (k-2)^2 - 64$.

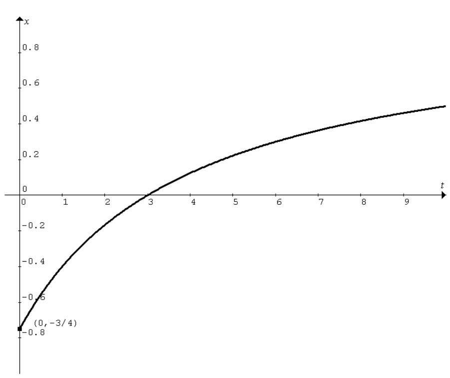
Q7cii $2x^2 + kx + 9 = 2x + 1$, $2x^2 + (k-2)x + 8 = 0$.
No intersections: $\Delta < 0$, $(k-2)^2 - 64 < 0$,
i.e. $(k-10)(k+6) < 0$, $\therefore -6 < k < 10$.

Q8ai $x = 1 - \frac{7}{t+4}$. When $t = 0$, $x = -\frac{3}{4}$.

Q8aii $v = \frac{dx}{dt} = \frac{7}{(t+4)^2}$. When $x = 0$, $1 - \frac{7}{t+4} = 0$, $t = 3$,
$v = \frac{7}{7^2} = \frac{1}{7}$.

Q8aiii $a = \frac{dv}{dt} = -\frac{14}{(t+4)^3} < 0$ for $t \geq 0$.

Q8aiv



Q8bi
$A_n = 200000r^n - M(1 + r + r^2 + ... + r^{n-1}) = 200000r^n - \frac{M(r^n - 1)}{r-1}$

$r = 1.006$, $A_{300} = 0$, $200000(1.006)^{300} - \frac{M(1.006^{300} - 1)}{1.006 - 1} = 0$,
$M = \$1439.18$

Q8bii $200000(1.006)^n - \frac{2800(1.006^n - 1)}{1.006 - 1} = 0$.
Expand and simplify:
$0.006(200000)(1.006)^n - 2800(1.006)^n + 2800 = 0$,
$1600(1.006)^n = 2800$, $n = 93.55$, i.e. 94 months.

Q9a Express $12y = x^2 - 6x - 3$ in the form $4a(y-h) = (x-k)^2$:
$12y = x^2 - 6x + 9 - 9 - 3$, $12y + 12 = x^2 - 6x + 9$,
$12(y+1) = (x-3)^2$, $\therefore a = 3$.
The vertex is $(h,k) = (3,-1)$, the focus is $(h, k+a) = (3,2)$.

Q9bi $\frac{dV}{dt} = 120 + 26t - t^2$. When $t = 0$, $\frac{dV}{dt} = 120$.
When $\frac{dV}{dt} = 2 \times 120$, $240 = 120 + 26t - t^2$, $t^2 - 26t + 120 = 0$,
$(t-6)(t-20) = 0$. Hence $t = 6$ or $t = 20$ min.

Q9bii $\Delta V(t) = \int_0^t (120 + 26t - t^2)dt = \left[120t + 13t^2 - \frac{t^3}{3}\right]_0^t$

$= 120t + 13t^2 - \frac{t^3}{3}$.

Q9biii When the tank is just full, $\Delta V = 7000 - 1500 = 5500$ L.
When $t = 30$, $\Delta V = 120(30) + 13(30)^2 - \frac{30^3}{3} = 6300$ L.
Loss $= 6300 - 5500 = 800$ L.

Q9ci Cone height $x = a + \sqrt{a^2 - r^2}$, $(x-a)^2 = a^2 - r^2$,
$\therefore r^2 = a^2 - (x-a)^2$, where $x > 0$.
Cone volume $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left[a^2 - (x-a)^2\right]x$

$= \frac{\pi}{3}(2ax^2 - x^3)$.

Q9cii At turning points, $\frac{dV}{dx} = \frac{\pi}{3}(4ax - 3x^2) = 0$,
$\therefore x(4a - 3x) = 0$. Since $x > 0$, $\therefore 4a - 3x = 0$, $x = \frac{4a}{3}$.

x	a	$\frac{4a}{3}$	$2a$
$\frac{dV}{dx}$	+	0	–

$\therefore x = \frac{4a}{3}$ gives the maximum volume.

Q10a
$\int_{0.5}^{1.5} (\log_e x)^3 dx \approx \frac{0.5}{3}\left[(\log_e 0.5)^3 + 4(\log_e 1.0)^3 + (\log_e 1.5)^3\right]$
$= -0.044$

Q10bi $KL = 6 + x$, $KQ = 6 - x$, where $0 \leq x \leq 6$,
$\therefore QL^2 = KL^2 - KQ^2 = (6+x)^2 - (6-x)^2 = 24x$.

Q10bii $\angle LMN + \angle MLN = \angle KLQ + \angle MLN = 90^0$,

$\therefore \angle LMN = \angle KLQ$. Given $\angle MNL = \angle LQK = 90^0$,

$\therefore \angle MLN = \angle LKQ$. Hence $\Delta QKL, \Delta NLM$ are similar.

$\therefore \frac{LM}{KL} = \frac{NM}{QL}$, $\frac{y}{6+x} = \frac{12}{\sqrt{24x}}$,

$\therefore y = \frac{12(6+x)}{\sqrt{24x}} = \frac{\sqrt{6}(6+x)}{\sqrt{x}}$.

Q10biii

Area of ΔKLM $= A = \frac{1}{2} \times \frac{\sqrt{6}(6+x)}{\sqrt{x}} \times (6+x) = \frac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$.

Q10biv $12 \le y \le 13$, $12 \le \frac{\sqrt{6}(6+x)}{\sqrt{x}} \le 13$, where $0 \le x \le 6$.

Square the terms: $144 \le \frac{6(6+x)^2}{x} \le 169$.

Expand and simplify: $144x \le 216 + 72x + 6x^2 \le 169x$.

Split into two inequalities:

$144x \le 216 + 72x + 6x^2$ and $216 + 72x + 6x^2 \le 169x$

$0 \le 216 - 72x + 6x^2$ and $216 - 97x + 6x^2 \le 0$

$\therefore 0 \le 6(6-x)^2$ and $(8-3x)(27-2x) \le 0$, i.e. $\frac{8}{3} \le x \le \frac{27}{2}$.

Since $0 \le x \le 6$, $\therefore \frac{8}{3} \le x \le 6$.

Q10bv $A = \frac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$, where $\frac{8}{3} \le x \le 6$

Turning points: $\frac{dA}{dx} = \frac{4\sqrt{6}\sqrt{x}(6+x) - \frac{\sqrt{6}(6+x)^2}{\sqrt{x}}}{4x} = 0$,

$4\sqrt{6}\sqrt{x}(6+x) - \frac{\sqrt{6}(6+x)^2}{\sqrt{x}} = 0$,

$4\sqrt{6}x(6+x) - \sqrt{6}(6+x)^2 = 0$, $\sqrt{6}(6+x)(4x-(6+x)) = 0$,

$3\sqrt{6}(6+x)(x-2) = 0$, $x = -6$ or 2.

Both values are outside $\frac{8}{3} \le x \le 6$.

$\frac{dA}{dx}$ is positive for $x > 2$, $\therefore A(x)$ is an increasing function of x for $\frac{8}{3} \le x \le 6$. Hence minimum area of ΔKLM occurs when

$x = \frac{8}{3}$. Minimum $A = \frac{\sqrt{6}\left(6 + \frac{8}{3}\right)^2}{2\sqrt{\frac{8}{3}}} = \frac{169}{3}$ square units.

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