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Applications of functions and calculus

Curve sketching

Sketching the graph of a function requires the knowledge of the x and y -intercepts, the asymptotic behaviours and the position and nature of each stationary point.

The x -coordinate of a stationary point can be found by setting $\frac{dy}{dx} = 0$, or $f'(x) = 0$, then solve for x . The y -coordinate is found by substituting the x value in the equation of the function.

The nature of a stationary point can be determined by finding the value of $\frac{d^2y}{dx^2}$ on each side of the stationary point.

Left	Stationary point	Right	Nature
$\frac{dy}{dx} > 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0$	Local max.
$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$	Local min.
$\frac{dy}{dx} > 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$	Inflection pt.
$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0$	Inflection pt.

Note: A local maximum (local minimum) is not necessarily the maximum (minimum) point of the function. You need to compare it with other stationary points and end points in the domain.

Example 1 Sketch $y = x^3 - 3x$, $x \in [-2, 2]$.

End points: $x = -2$, $y = -2$; $x = 2$, $y = 2$.

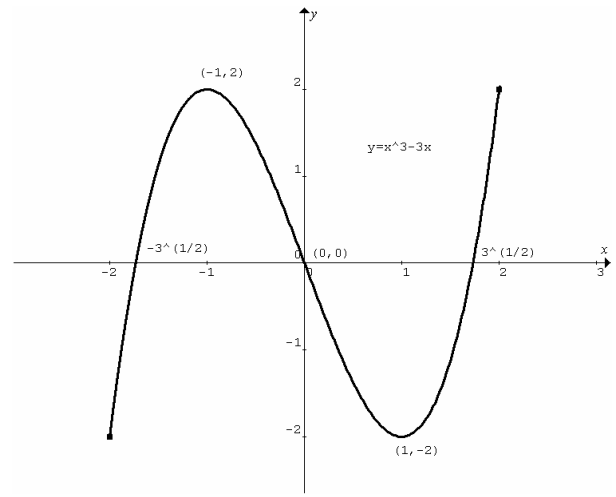
x -intercepts: Let $y = 0$, $x^3 - 3x = 0$, $x(x^2 - 3) = 0$,
 $x(x - \sqrt{3})(x + \sqrt{3}) = 0$, $x = -\sqrt{3}, 0, \sqrt{3}$.

Stationary points: Let $\frac{dy}{dx} = 0$, $\frac{dy}{dx} = 3(x^2 - 1) = 0$, $\therefore x = -1, 1$.
 $\therefore y = 2, -2$.

Nature of stationary points:

At $x =$	-2	-1	0	1	2
$\frac{dy}{dx}$	> 0	$= 0$	< 0	$= 0$	> 0
Nature		Local max.		Local min.	

Asymptotic behaviour: None.



Example 2 Sketch the graph of $x = \frac{1}{10}(t-5)^2 e^{-0.1t}$, $t \in [0, \infty)$.

t -intercepts: Let $x = 0$, $\frac{1}{10}(t-5)^2 e^{-0.1t} = 0$. Since

$e^{-0.1t} > 0$, $\therefore t-5 = 0$ or $t = 5$. This intercept is also a turning point because $(t-5)$ is a repeated factor.

x -intercept: Let $t = 0$, $x = 2.5$. This is also an end point.

Asymptotic behaviour: As $t \rightarrow \infty$, the dominant factor

$e^{-0.1t} \rightarrow 0^+$, $\therefore x \rightarrow 0^+$.

Stationary points: $\frac{dx}{dt} = 0$,

$$-0.01(t-5)^2 e^{-0.1t} + 0.2(t-5)e^{-0.1t} = 0,$$

$$-(t-5)e^{-0.1t}(0.01(t-5) - 0.2) = 0,$$

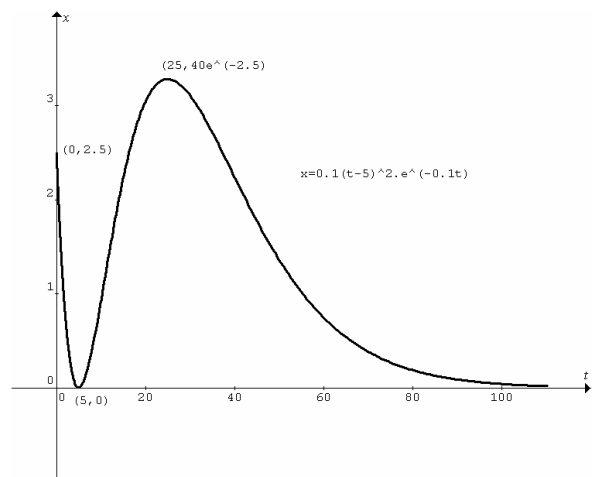
$$-(t-5)(0.01t - 0.25)e^{-0.1t} = 0. \text{ Since } e^{-0.1t} > 0,$$

$$\therefore t-5 = 0, \text{ i.e. } t = 5 \text{ and } x = 0 \text{ (as discussed earlier),}$$

$$\text{or } 0.01t - 0.25 = 0, \text{ i.e. } t = 25 \text{ and } x = 40e^{-2.5}.$$

Nature of stationary points:

At $t =$	4	5	10	25	26
$\frac{dx}{dt}$	< 0	$= 0$	> 0	$= 0$	< 0
Nature		Local min.		Local max.	



Example 3 Sketch $y = e^{\frac{x}{\sqrt{3}}} \cos(x)$ for $x \in [0, 2\pi]$.

End points: At $x = 0$, $y = 1$. At $x = 2\pi$, $y = e^{\frac{2\pi}{\sqrt{3}}}$.

x -intercepts: Let $y = 0$, $e^{\frac{x}{\sqrt{3}}} \cos(x) = 0$, and since $e^{\frac{x}{\sqrt{3}}} > 0$,

$\therefore \cos(x) = 0$, and hence $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Asymptotic behaviour: None.

Stationary points: $\frac{dy}{dx} = 0$, $-e^{\frac{x}{\sqrt{3}}} \sin(x) + \frac{1}{\sqrt{3}} e^{\frac{x}{\sqrt{3}}} \cos(x) = 0$,

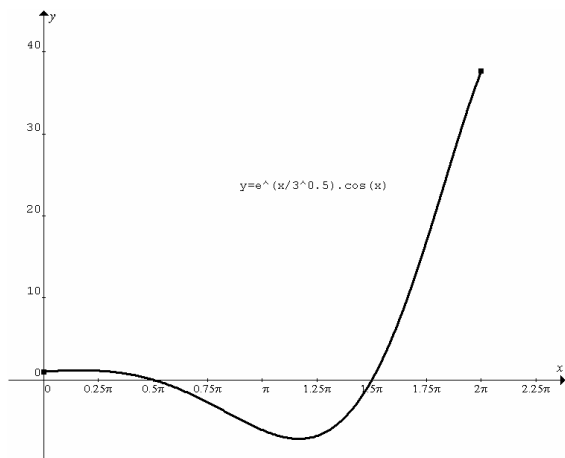
$-e^{\frac{x}{\sqrt{3}}} \left(\sin(x) - \frac{1}{\sqrt{3}} \cos(x) \right) = 0$, $\therefore \sin(x) = \frac{1}{\sqrt{3}} \cos(x)$,

$\tan(x) = \frac{1}{\sqrt{3}}$, hence $x = \frac{\pi}{6}$ and $y = \frac{\sqrt{3}}{2} e^{\frac{\pi}{6\sqrt{3}}}$ or $x = \frac{7\pi}{6}$ and

$y = -\frac{\sqrt{3}}{2} e^{\frac{7\pi}{6\sqrt{3}}}$.

Nature of stationary points:

At $x =$	0.5	$\frac{\pi}{6}$	1	$\frac{7\pi}{6}$	4
$\frac{dy}{dx}$	> 0	$= 0$	< 0	$= 0$	> 0
Nature		Local max.		Local min.	



Example 4 Sketch $y = \frac{1}{x+2} + \frac{1}{3-x} + 1$ for $x \in [-1, 3)$.

End point: At $x = -1$, $y = 2.25$.

Asymptotic behaviour: As $x \rightarrow 3^-$, $y \rightarrow \infty$.

The function is positive for $x \in [-1, 3)$, no x -intercepts.

y -intercept: Let $x = 0$, $y = \frac{11}{6}$.

Stationary points: $\frac{dy}{dx} = -\frac{1}{(x+2)^2} + \frac{1}{(3-x)^2} = 0$,

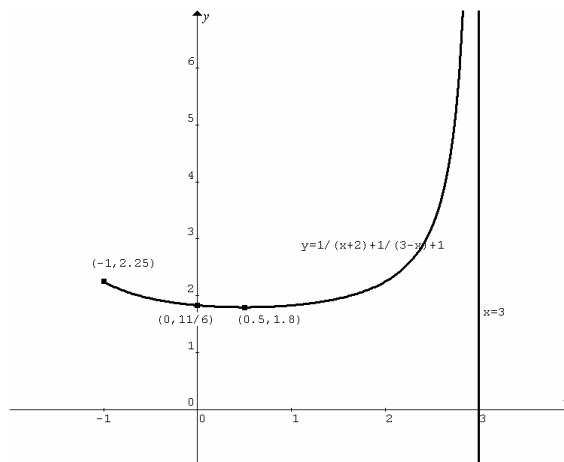
$\frac{(x+2)^2 - (3-x)^2}{(3-x)^2(x+2)^2} = 0$, $\therefore (x+2)^2 - (3-x)^2 = 0$,

$((x+2) - (3-x))((x+2) + (3-x)) = 0$, $5(2x-1) = 0$, $x = \frac{1}{2}$ and

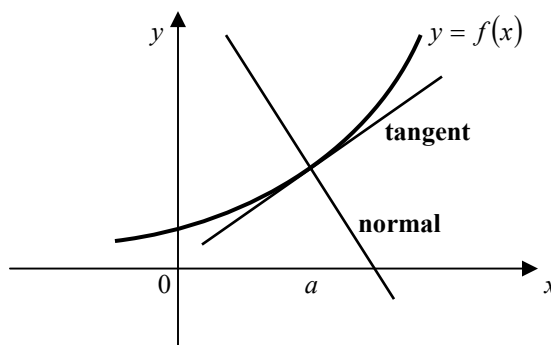
$y = \frac{9}{5}$.

Nature of stationary point:

At $x =$	0	$\frac{1}{2}$	1
$\frac{dy}{dx}$	< 0	$= 0$	> 0
Nature		Local min.	



Equations of tangents and normals



Gradient of the tangent to the curve $y = f(x)$ at $x = a$ is $m_T = f'(a)$.

Gradient of the normal is $m_N = -\frac{1}{m_T} = -\frac{1}{f'(a)}$.

Use $y - y_1 = m(x - x_1)$ to find equations of tangents and normals.

For tangents: $y - f(a) = f'(a)(x - a)$.

For normals: $y - f(a) = -\frac{1}{f'(a)}(x - a)$.

Example 1 Find the equation of the normal to the curve $y = 3 \log_e(2x+1) - 1$ at $x = 0$.

At $x = 0$, $y = -1$, $m_T = \frac{dy}{dx} = \frac{6}{2x+1} = 6$ and $\therefore m_N = -\frac{1}{6}$.

Equation of the normal: $y - (-1) = -\frac{1}{6}(x - 0)$, $\therefore y = -\frac{1}{6}x - 1$.

Example 2 Find the equation of the tangent to the curve $y = a(x+1)(x-1)^2$ at the y -intercept. Find the point where the tangent crosses the curve. Explain why this point is independent of a .

At the y -intercept, $x = 0$, $y = a$,

$$\begin{aligned} \frac{dy}{dx} &= 2a(x+1)(x-1) + a(x-1)^2 = a(x-1)(2(x+1) + x-1) \\ &= a(x-1)(3x+1) = -a \end{aligned}$$

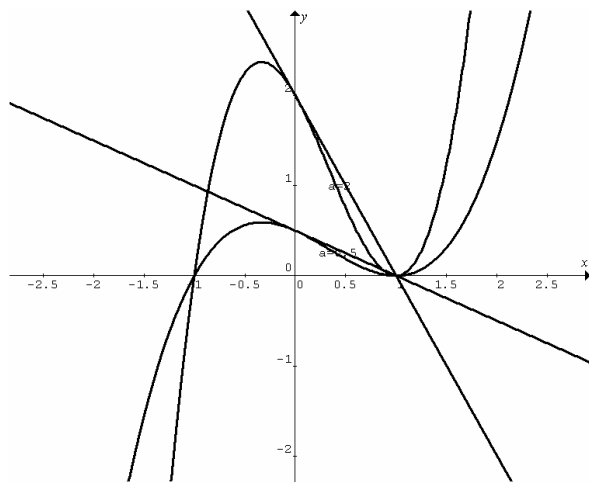
Equation of the tangent at the y -intercept: $y - a = -a(x - 0)$,
i.e. $y = -a(x - 1)$.

Solve $y = -a(x - 1)$ and $y = a(x + 1)(x - 1)^2$ simultaneously to find the intersection. $\therefore -a(x - 1) = a(x + 1)(x - 1)^2$,

$$\therefore a(x - 1) + a(x + 1)(x - 1)^2 = 0, \quad a(x - 1)(1 + (x + 1)(x - 1)) = 0,$$

$\therefore a(x - 1)x^2 = 0$, $\therefore x = 0$ (where the line touches the curve) or $x = 1$ (where the line crosses the curve) and $y = 0$.

The intersecting point is $(1, 0)$, and since the parameter a does not appear in the coordinates, it is independent of a .



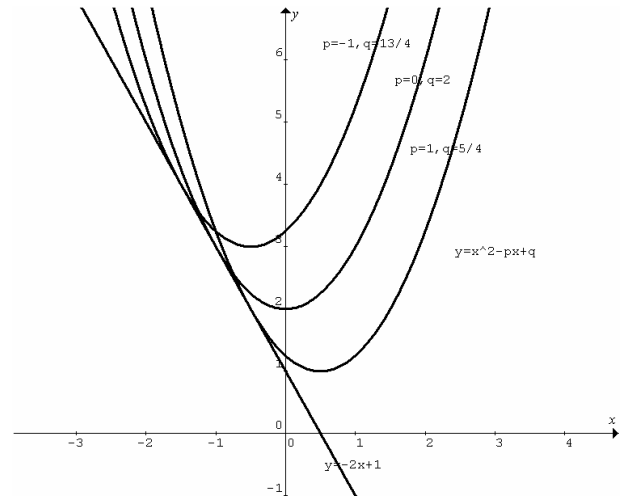
Example 3 The line $y = -2x + 1$ is a tangent to the parabola $y = x^2 - px + q$. Find the values of p and q .

$m_T = \frac{dy}{dx} = 2x - p = -2$. $\therefore x = \frac{p-2}{2}$ is the x -coordinate of the point of contact. The y -coordinate is found by substituting

$$x = \frac{p-2}{2} \text{ in } y = -2x + 1, \therefore y = 3 - p,$$

$$\text{or in } y = x^2 - px + q, \therefore y = \left(\frac{p-2}{2}\right)^2 - \frac{p(p-2)}{2} + q.$$

$\therefore 3 - p = \left(\frac{p-2}{2}\right)^2 - \frac{p(p-2)}{2} + q$, it can be simplified to $(2 - p)^2 = 4(q - 1)$. The values of p and q have to satisfy this relationship, e.g. if $p = 0$, $q = 2$; if $p = 1$, $q = \frac{5}{4}$; if $p = -1$, $q = \frac{13}{4}$. The number of possibilities is infinite.

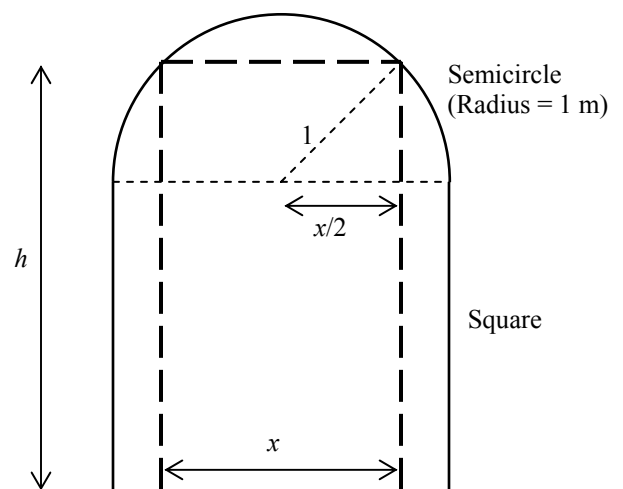


Maximum/minimum problems

In practical situations, the value of a quantity (dependent variable) varies with the value of another quantity (independent variable). For certain value of the latter, the value of the former is a maximum/minimum. This can be found by differentiation or graphics calculator.

In formulating a problem, sometimes the dependent variable is a function of two (or more) other variables. It is necessary to find the relationship between the two other variables. The dependent variable can then be expressed in terms of one of them before finding the maximum/minimum.

Example 1 An upright rectangular window is to be installed in the middle of a wall with dimensions shown in the following figure.



Find the area of the largest window possible for the wall.

Let x m and h m be the width and the height of the window respectively.

The area of the window $A = xh$.

The relationship between h and x is $h = 2 + \sqrt{1 - \left(\frac{x}{2}\right)^2}$.

$$\therefore A(x) = x \left(2 + \sqrt{1 - \left(\frac{x}{2}\right)^2} \right) = x \left(2 + \sqrt{1 - \frac{x^2}{4}} \right).$$

Use graphics calculator to graph the area function $A(x)$ and determine the maximum.

$$x = 1.86 \text{ m}, A = 4.40 \text{ m}^2.$$

Example 2 Find the points on the curve $y = x^2 + 2x - 1$ that are closest to the point $(-1, -1)$.

Let (x, y) be a point on the curve closest to $(-1, -1)$. The distance between the two points is:

$$D(x) = \sqrt{(x - (-1))^2 + (y - (-1))^2} = \sqrt{(x + 1)^2 + (x^2 + 2x)^2}.$$

$$\frac{dD}{dx} = \frac{1}{2\sqrt{(x+1)^2 + (x^2+2x)^2}} (2(x+1) + 2(x^2+2x)(2x+2)).$$

$$\text{Let } \frac{dD}{dx} = 0, \therefore 2(x+1) + 2(x^2+2x)(2x+2) = 0,$$

$$2(x+1)(1+2(x^2+2x)) = 0, \therefore x = -1 \text{ and } y = -2 \text{ or}$$

$$2x^2 + 4x + 1 = 0, \text{ i.e. } x = \frac{-4 + \sqrt{16-8}}{4} = \frac{-2 + \sqrt{2}}{2} \text{ and}$$

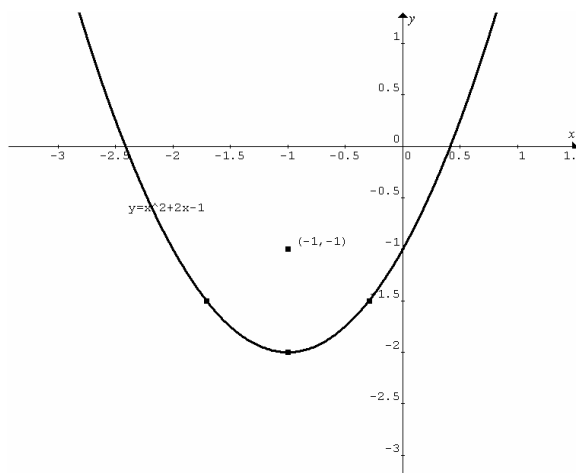
$$y = \left(\frac{-2 + \sqrt{2}}{2}\right)^2 + 2\left(\frac{-2 + \sqrt{2}}{2}\right) - 1 = \frac{3-2\sqrt{2}}{2} - 2 + \sqrt{2} - 1 = -\frac{3}{2}$$

$$\text{or } x = \frac{-4 - \sqrt{16-8}}{4} = \frac{-2 - \sqrt{2}}{2} \text{ and}$$

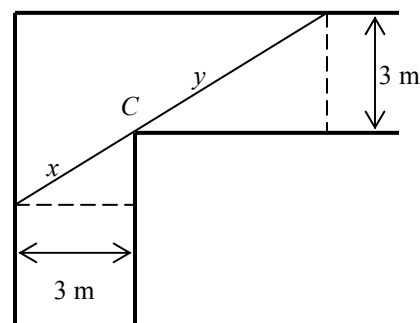
$$y = \left(\frac{-2 - \sqrt{2}}{2}\right)^2 + 2\left(\frac{-2 - \sqrt{2}}{2}\right) - 1 = \frac{3+2\sqrt{2}}{2} - 2 - \sqrt{2} - 1 = -\frac{3}{2}.$$

The three points closest to $(-1, -1)$ are $(-1, -2)$,

$$\left(\frac{-2 + \sqrt{2}}{2}, -\frac{3}{2}\right) \text{ and } \left(\frac{-2 - \sqrt{2}}{2}, -\frac{3}{2}\right).$$



Example 3 Find the length of the longest rigid pipe that can be carried horizontally from a 3-m wide corridor to another 3-m wide corridor perpendicular to the first. Ignore the diameter of the pipe.



Let the horizontal straight line distance through the corner C from one inside wall to the next wall be $L = x + y$ metres. There is a relationship between x and y . Similar triangles,

$$\frac{y}{x} = \frac{3}{\sqrt{x^2 - 9}}, \therefore y = \frac{3x}{\sqrt{x^2 - 9}}, \therefore L(x) = x + \frac{3x}{\sqrt{x^2 - 9}}.$$

The longest possible pipe is restricted by the shortest distance L .

$$\frac{dL}{dx} = 1 + \frac{3\sqrt{x^2 - 9} - (3x)\left(\frac{x}{\sqrt{x^2 - 9}}\right)}{x^2 - 9}$$

$$= 1 + \frac{3(x^2 - 9) - 3x^2}{(x^2 - 9)\sqrt{x^2 - 9}} = 1 - \frac{27}{(x^2 - 9)^{\frac{3}{2}}}. \text{ Let } \frac{dL}{dx} = 0,$$

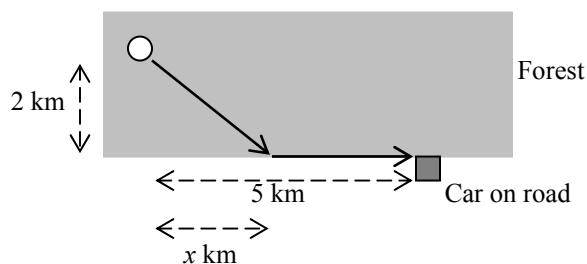
$$1 - \frac{27}{(x^2 - 9)^{\frac{3}{2}}} = 0, \frac{27}{(x^2 - 9)^{\frac{3}{2}}} = 1, (x^2 - 9)^{\frac{1}{2}} = 3, x^2 - 9 = 9,$$

$$\therefore x = 3\sqrt{2} \text{ and } y = \frac{9\sqrt{2}}{3} = 3\sqrt{2}. \text{ Hence } L = 6\sqrt{2}.$$

$x =$	4	$3\sqrt{2}$	5
$\frac{dL}{dx}$	< 0	$= 0$	> 0
Nature		Local min.	

$\therefore L = 6\sqrt{2}$ is the shortest distance and hence the longest pipe is $6\sqrt{2}$ m.

Example 4 A bushwalker is in a forest 2 km from a straight road. His car is 5 km down the road. He can walk 6 kmh^{-1} in the forest and 8 kmh^{-1} along the road. Toward what point on the road should he walk to minimise the time required to reach his car?



Let x km be the distance of the bushwalker from the point along the road. He walks $\sqrt{x^2 + 2^2}$ km in the forest and $5 - x$ km along the road. Using the formula $\text{time} = \frac{\text{distance}}{\text{speed}}$, the total

$$\text{time taken } T(x) = \frac{\sqrt{x^2 + 4}}{6} + \frac{5 - x}{8} \text{ hours, where } 0 \leq x \leq 5.$$

$$T'(x) = \frac{1}{6 \times 2\sqrt{x^2 + 4}} \times 2x - \frac{1}{8} = \frac{x}{6\sqrt{x^2 + 4}} - \frac{1}{8}.$$

$$\text{Let } T'(x) = 0, \frac{x}{6\sqrt{x^2 + 4}} - \frac{1}{8} = 0, 6\sqrt{x^2 + 4} = 8x,$$

$$3\sqrt{x^2 + 4} = 4x, 9(x^2 + 4) = 16x^2, 7x^2 = 36,$$

$$\therefore x = \frac{6}{\sqrt{7}} = \frac{6\sqrt{7}}{7}.$$

$x =$	2	$\frac{6\sqrt{7}}{7}$	3
$\frac{dT}{dx}$	< 0	$= 0$	> 0
Nature		Local min.	

\therefore the point that minimise the time is $\frac{6\sqrt{7}}{7}$ km along the road from the initial position of the bushwalker.

Example 5 The concentration y of a drug in the blood time t after injection is given by $y = \frac{k}{a-b}(e^{-bt} - e^{-at})$, where a, b and k are positive constants and $a > b$. Find the maximum concentration and the time it occurs after injection.

$$\frac{dy}{dt} = \frac{k}{a-b}(-be^{-bt} + ae^{-at}). \text{ Let } \frac{dy}{dt} = 0,$$

$$\frac{k}{a-b}(-be^{-bt} + ae^{-at}) = 0, \therefore -be^{-bt} + ae^{-at} = 0, \frac{e^{-bt}}{e^{-at}} = \frac{a}{b},$$

$$\therefore e^{at-bt} = \frac{a}{b}, e^{(a-b)t} = \frac{a}{b}, (a-b)t = \log_e\left(\frac{a}{b}\right),$$

$$t = \frac{1}{a-b} \log_e\left(\frac{a}{b}\right).$$

$$\text{At } t < \frac{1}{a-b} \log_e\left(\frac{a}{b}\right), (a-b)t < \log_e\left(\frac{a}{b}\right), e^{(a-b)t} < \frac{a}{b},$$

$$\frac{e^{-bt}}{e^{-at}} < \frac{a}{b}, \therefore be^{-bt} - ae^{-at} < 0, \therefore -be^{-bt} + ae^{-at} > 0, \therefore \frac{dy}{dt} > 0.$$

$$\text{Similarly, at } t > \frac{1}{a-b} \log_e\left(\frac{a}{b}\right), \frac{dy}{dt} < 0.$$

$$\therefore \text{at } t = \frac{1}{a-b} \log_e\left(\frac{a}{b}\right), y \text{ is maximum.}$$

$$y_{\max} = \frac{k}{a-b} \left(e^{-\frac{b}{a-b} \ln\left(\frac{a}{b}\right)} - e^{-\frac{a}{a-b} \ln\left(\frac{a}{b}\right)} \right).$$

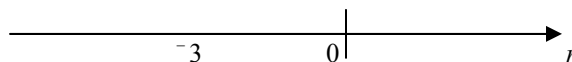
Rate of change of a function

If $f(x)$ is any function differentiable at $x = a$, then the rate of change of $f(x)$ with respect to x at $x = a$ is $f'(a)$.

The average rate of change of $f(x)$ with respect to x between $x = a$ and $x = b$, where $b > a$, is $\frac{f(b) - f(a)}{b - a}$.

Example 1 The position of a particle moving along a straight line at time t is $r(t) = t^2 - 2t - 3$, where $0 \leq t \leq 6$. Given the initial velocity $v(0) = -2$, find (a) the initial position of the particle, (b) the rate of change of position with respect to time (i.e. the velocity) at $t = 1$, (c) the average rate of change of position with respect to time (i.e. average velocity) between $t = 0$ and $t = 5$, (d) the rate of change of velocity with respect to time (i.e. the acceleration) at $t = 1$, (e) the average rate of change of velocity with respect to time (i.e. the average acceleration) between $t = 0$ and $t = 5$, (f) sketch the r - t graph, (g) use the graph to find the distance travelled from $t = 0$ to $t = 5$, (h) the average rate of change in distance travelled with respect to time (i.e. the average speed) between $t = 0$ and $t = 5$.

(a) Initial position: at $t = 0$, $r(0) = -3$.



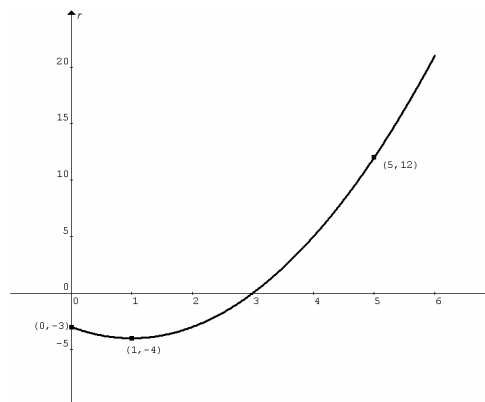
(b) Velocity: $v(t) = \frac{dr}{dt} = 2t - 2$, at $t = 1$, $v(1) = 0$, i.e. the particle is momentarily at rest.

(c) Average velocity: $v_{av} = \frac{r(5) - r(0)}{5 - 0} = \frac{+12 - (-3)}{5} = +3$

(d) Acceleration: $a = \frac{dv}{dt} = +2$, which is a constant. Same acceleration for $t > 0$.

(e) Average acceleration: $a_{av} = \frac{v(5) - v(0)}{5 - 0} = \frac{+8 - (-2)}{5} = +2$. a and a_{av} are the same because a is constant.

(f)



(g) The particle moves backwards from -3 to -4 , then forwards to $+12$. Total distance = $1 + 16 = 17$.

(h) Average speed = $\frac{\text{distance}}{\text{time}} = \frac{17}{5} = 3.4$, which is different from the average velocity.

Example 2 (a) Find the rate of change of volume V of a spherical balloon with respect to its radius r when $r = 5$.
 (b) Find the rate of change of volume V of the spherical balloon with respect to its surface area A when $r = 5$.

(a) Volume of a sphere $V(r) = \frac{4}{3}\pi r^3$. $\frac{dV}{dr} = 4\pi r^2$, when $r = 5$, $\frac{dV}{dr} = 4\pi(5^2) = 100\pi$.

(b) Surface area $A = 4\pi r^2$, $\therefore r = \sqrt{\frac{A}{4\pi}}$,
 $\therefore V(A) = \frac{4\pi}{3} \left(\sqrt{\frac{A}{4\pi}} \right)^3 = \frac{1}{3\sqrt{4\pi}} A^{\frac{3}{2}} = \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}}$.

$\frac{dV}{dA} = \frac{3}{2} \times \frac{1}{6\sqrt{\pi}} A^{\frac{1}{2}} = \frac{1}{4\sqrt{\pi}}$, when $r = 5$, $A = 100\pi$,
 $\therefore \frac{dV}{dA} = \frac{5}{2}$.

Note: (b) can be done in the following way since the two rates, $\frac{dV}{dr}$ and $\frac{dV}{dA}$ are related.

$V(r) = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2$. $A(r) = 4\pi r^2$, $\frac{dA}{dr} = 8\pi r$.

Using the chain rule: $\frac{dV}{dr} = \frac{dV}{dA} \times \frac{dA}{dr}$,
 $\therefore \frac{dV}{dA} = \frac{dV}{dr} \bigg/ \frac{dA}{dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$. When $r = 5$, $\frac{dV}{dA} = \frac{5}{2}$.

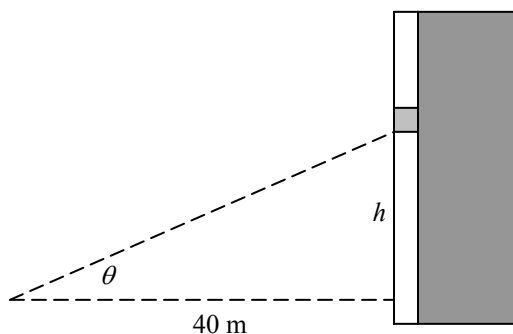
Related rates

If two quantities x and y are related by $y = f(x)$, then the rate of change of y with respect to a third quantity t is related to the rate of change of x with respect to the third quantity t by

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ or $\frac{dy}{dt} = f'(x) \times \frac{dx}{dt}$.

Example 1 At a distance of 40 m from the base of a tall building, a person observes an external elevator moving up at a constant speed of 2.0 ms^{-1} . Determine the rate of change of the angle of elevation when the elevator is 30 m above the person's eye level.

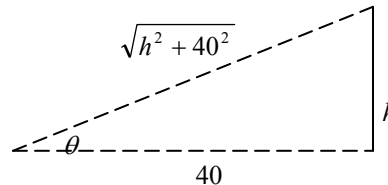
The relationship between the height above the eye level and the angle of elevation is $h = 40 \tan \theta$.



$\frac{dh}{d\theta} = 40 \sec^2 \theta$. (Note: the derivative is correct only when θ is in radians)

The chain rule: $\frac{dh}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dt}$,
 $\therefore \frac{d\theta}{dt} = \frac{dh}{dt} \bigg/ \frac{dh}{d\theta} = \frac{2.0}{40 \sec^2 \theta} = \frac{1}{20} \cos^2 \theta$.

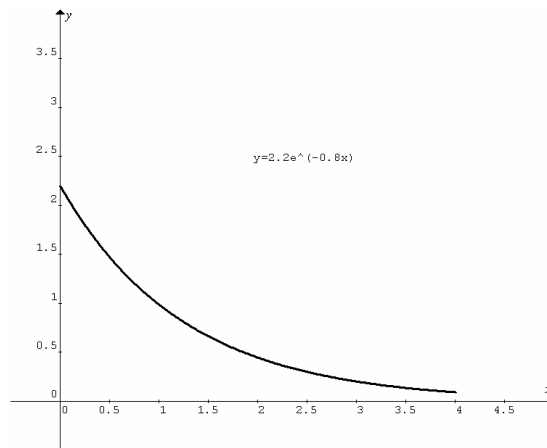
Instead of finding θ when $h = 30$, and substitute into the above expression, find $\cos \theta$ in terms of h .



$\therefore \cos \theta = \frac{40}{\sqrt{h^2 + 40^2}}$. $\therefore \frac{d\theta}{dt} = \frac{1}{20} \left(\frac{40}{\sqrt{h^2 + 40^2}} \right)^2$.

When $h = 30$, $\frac{d\theta}{dt} = 0.032$ radians per second
 $= 0.032 \times \frac{180^\circ}{\pi} \approx 1.8$ degrees per second.

Example 2 The side elevation of a playground slide can be modeled by $y = 2.2e^{-0.8x}$ for $0 \leq x \leq 4$. Length measures are in metres and time in seconds. The shadow on the ground of a girl sliding down moves at 1.5 ms^{-1} when she is at $x = 2.0$. Calculate her descending speed (vertical) at that moment.



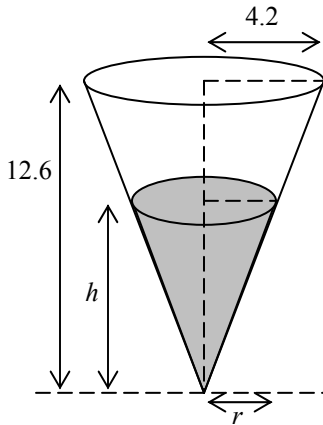
$y = 2.2e^{-0.8x}$, $\frac{dy}{dx} = -1.76e^{-0.8x}$.

The chain rule: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

Given the velocity of the shadow $= \frac{dx}{dt} = 1.5$ at $x = 2.0$,

the vertical velocity $= \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$
 $= -1.76e^{-0.8(2.0)} \times 1.5 = -0.53 \text{ ms}^{-1}$. The negative sign indicates downward motion.

Example 3 Water is poured into a conical cup at a rate of 5.0 cm^3 per second. The cup is 12.6 cm tall and the radius of the rim of the cup is 4.2 cm . How fast does the water level rise when the depth of water is 8.0 cm ?



Use similar triangles to find the relationship between r and h .

$$\frac{r}{4.2} = \frac{h}{12.6}, \therefore r = \frac{h}{3}$$

Volume of a cone = $\frac{1}{3}\pi R^2 H$, \therefore volume of water = $V = \frac{1}{3}\pi r^2 h$.

Substitute $r = \frac{h}{3}$ into the last expression, $V(h) = \frac{\pi h^3}{27}$, and

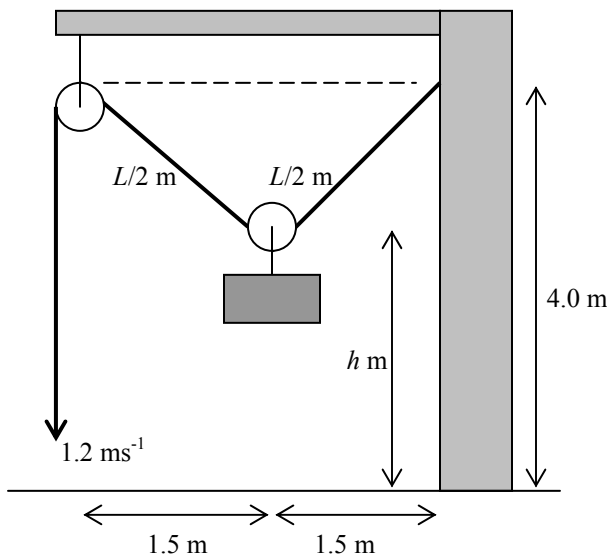
$$\frac{dV}{dh} = \frac{\pi h^2}{9}$$

The chain rule: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$, $\therefore \frac{dh}{dt} = \frac{dV}{dt} / \frac{dV}{dh}$.

Given $\frac{dV}{dt} = 5.0$ (constant), at $h = 8.0$, the rate of change of

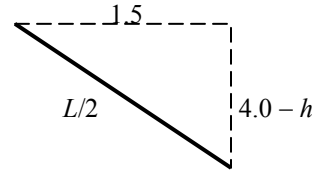
$$\text{depth} = \frac{dh}{dt} = \frac{5.0}{\frac{\pi(8.0^2)}{9}} = 0.22 \text{ cm.}$$

Example 4 A load is lifted upwards by a person pulling a rope at 1.2 ms^{-1} . See the diagram below. How fast does the load rise when the lower pulley is 2.0 m above the ground?



Let $L \text{ m}$ be the length of the rope from the upper pulley through the lower pulley to the wall and $h \text{ m}$ be the height of the lower pulley above the ground. Assume zero size for the pulleys.

Use Pythagorean Theorem to find the relationship between L and h .



$$\left(\frac{L}{2}\right)^2 = 1.5^2 + (4-h)^2, \therefore L = 2\sqrt{2.25 + (4-h)^2},$$

$$\frac{dL}{dh} = \frac{1}{\sqrt{2.25 + (4-h)^2}} \times (-2(4-h)) = \frac{-2(4-h)}{\sqrt{2.25 + (4-h)^2}}$$

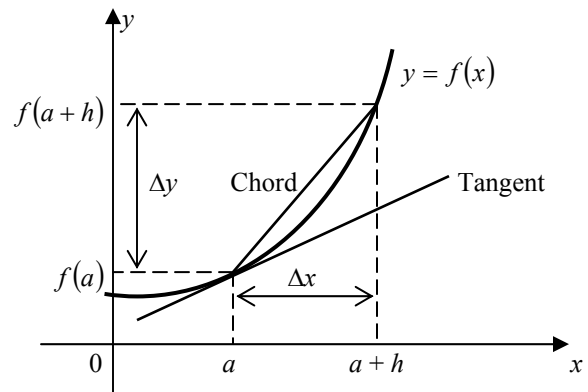
The chain rule: $\frac{dL}{dt} = \frac{dL}{dh} \times \frac{dh}{dt}$, $\therefore \frac{dh}{dt} = \frac{dL}{dt} / \frac{dL}{dh}$.

Given $\frac{dL}{dt} = -1.2$ (constant, negative because L is decreasing),

when $h = 2.0$, the raising speed of the load

$$= \frac{dh}{dt} = \frac{-1.2}{\frac{-2(4-2)}{\sqrt{2.25 + (4-2)^2}}} = 0.75 \text{ ms}^{-1}.$$

Linear approximation $f(a+h) \approx f(a) + hf'(a)$



Gradient of the tangent \approx gradient of the chord, this approximation is good for small h ,

$$\text{i.e. } \left. \frac{dy}{dx} \right|_{x=a} \approx \frac{\Delta y}{\Delta x} \text{ or } f'(a) \approx \frac{f(a+h) - f(a)}{h},$$

$$\therefore \Delta y \approx \Delta x \left. \frac{dy}{dx} \right|_{x=a} \text{ or } f(a+h) \approx f(a) + hf'(a).$$

Example 1 Given $\frac{dy}{dx} = 2e^x$, find the change in y when x changes from 1 to 1.1.

$$\Delta y \approx \Delta x \frac{dy}{dx} \Big|_{x=a} = (1.1-1) \frac{dy}{dx} \Big|_{x=1} = 0.1 \times 2e = 0.2e.$$

Example 2 Find the approximate increase in radius when the volume of a spherical balloon increases from 1.0 m^3 to 1.1 m^3 .

$$V = \frac{4}{3}\pi r^3, \quad r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} V^{\frac{1}{3}},$$

$$\frac{dr}{dV} = \frac{1}{3} \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} V^{-\frac{2}{3}} = \left(\frac{1}{36\pi}\right)^{\frac{1}{3}} V^{-\frac{2}{3}}.$$

$$\Delta r \approx \Delta V \frac{dr}{dV} \Big|_{V=1.0} = 0.1 \left(\frac{1}{36\pi}\right)^{\frac{1}{3}} (1.0)^{-\frac{2}{3}} = 0.1 \left(\frac{1}{36\pi}\right)^{\frac{1}{3}}.$$

Example 3 Find the approximate value of $\sqrt{101}$.

$$f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}}, \quad \text{let } x = 100, \text{ then } h = 1,$$

$$f(100) = \sqrt{100} = 10, \quad f'(100) = \frac{1}{2\sqrt{100}} = 0.05.$$

Since $f(a+h) \approx f(a) + hf'(a)$, $\therefore \sqrt{101} \approx 10 + 1 \times 0.05 = 10.05$

Example 4 Given that $f(3) = 0.01$ and $f'(3) = -2$, find the approximate solution (close to 3) of the equation $f(x) = 0$.

Using $f(a+h) \approx f(a) + hf'(a)$, $0 \approx 0.01 + h(-2)$, $\therefore h \approx 0.005$.

Hence the approximate solution is

$$x = a + h = 3 + 0.005 = 3.005.$$

Example 5 Given $f(x) = x^3 - 3x + 3$, find $f(-2.1)$ and hence find an approximate zero of $f(x)$.

$$f(-2.1) = 0.039 \text{ that is close to zero. } f'(x) = 3x^2 - 3,$$

$$\therefore f'(-2.1) = 10.23.$$

Using $f(a+h) \approx f(a) + hf'(a)$, $0 \approx 0.039 + h(10.23)$,

$$\therefore h \approx -0.0038. \text{ Hence an approximate zero is}$$

$$x = a + h = -2.1 - 0.0038 = -2.1038.$$

Example 6 The formula for calculating the average speed is

$v = \frac{D}{t}$. If there is a 5% error in measuring the time t , what is the percentage error in the calculated average speed?

$$v = \frac{D}{t}, \quad \frac{\Delta v}{\Delta t} \approx \frac{dv}{dt}, \quad \therefore \frac{\Delta v}{\Delta t} \approx -\frac{D}{t^2}, \quad \therefore \frac{\Delta v}{\Delta t} \approx -\frac{v}{t}, \quad \therefore \frac{\Delta v}{v} \approx -\frac{\Delta t}{t},$$

$$\% \text{ error in } v = \frac{\Delta v}{v} \times 100\% \approx -\frac{\Delta t}{t} \times 100\% = -5\% \text{ (The negative}$$

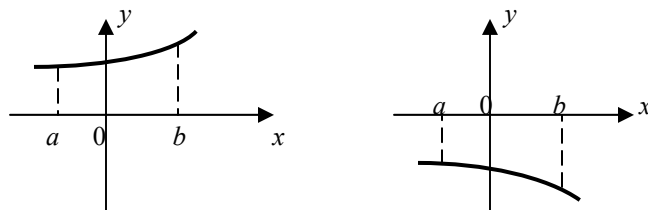
sign indicates that the calculated v is lower than the true v if the measured t is higher than the true t)

Finding area under a curve by integration

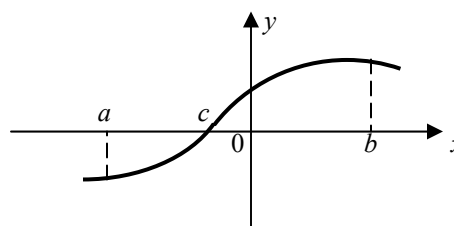
The area of the region bounded by the x -axis and a curve

$y = f(x)$ in the interval $[a, b]$ is given by $A = \left| \int_a^b f(x) dx \right|$ if

$f(x) \geq 0$ for all $x \in [a, b]$, or $f(x) \leq 0$ for all $x \in [a, b]$.



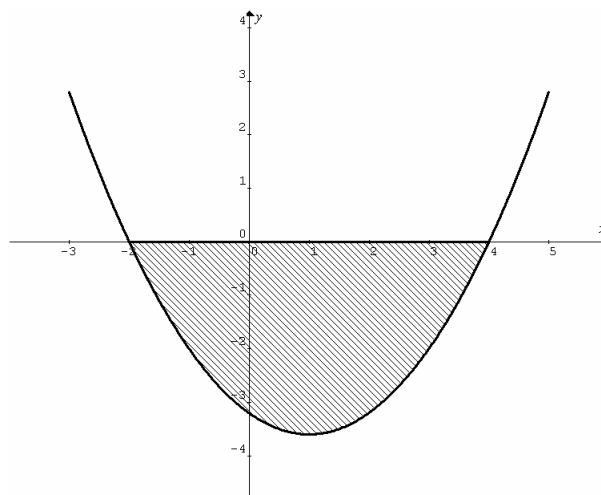
If $f(x)$ changes sign at $x = c$ in the interval $[a, b]$, then two definite integrals are required to evaluate the area of the bounded region, $A = \left| \int_a^c f(x) dx \right| + \left| \int_c^b f(x) dx \right|$.



If $f(x)$ changes sign again, a third definite integral is required.

Note: It is important to know the behaviour (graph) of the function before setting up the definite integral(s) to find the area.

Example 1 Find the area of the region enclosed by the x -axis and $y = 0.4(x+2)(x-4)$.



The left and right extremes of the region are the x -intercepts, $x = -2$ and $x = 4$ respectively, given by the linear factors.

Expand $y = 0.4(x+2)(x-4)$ to obtain $y = 0.4(x^2 - 2x - 8)$.

$$\begin{aligned} \text{Area} &= \left| \int_{-2}^4 0.4(x^2 - 2x - 8) dx \right| = \left| \left[0.4 \left(\frac{x^3}{3} - x^2 - 8x \right) \right]_{-2}^4 \right| \\ &= \left| 0.4 \left(\frac{4^3}{3} - 4^2 - 8(4) \right) - 0.4 \left(\frac{(-2)^3}{3} - (-2)^2 - 8(-2) \right) \right| = 14.4 \end{aligned}$$

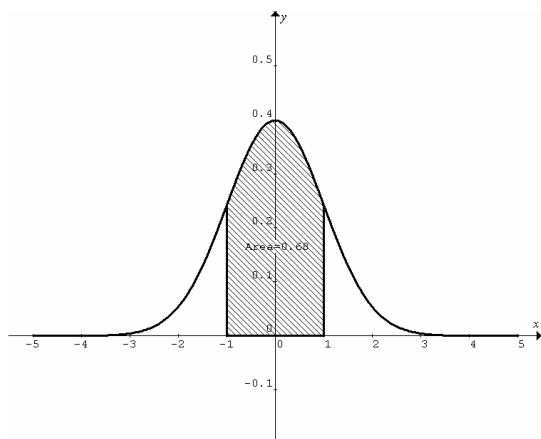
Example 2 Consider the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. This is a probability density function known as the standard normal distribution. The area under the curve from $x = a$ to $x = b$ gives the probability that x lies between a and b ,

i.e. $\Pr(a < x < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. Use graphics calculator to

evaluate the following probabilities.

- (a) $\Pr(-1 < x < 1)$, (b) $\Pr(-2 < x < 2)$, (c) $\Pr(-3 < x < 3)$,
 (d) $\Pr(-\infty < x < \infty)$.

Sketch $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, 2nd calc $\int f(x)dx$ etc.



(a) $\Pr(-1 < x < 1) = \int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0.68$

(b) $\Pr(-2 < x < 2) = \int_{-2}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0.95$

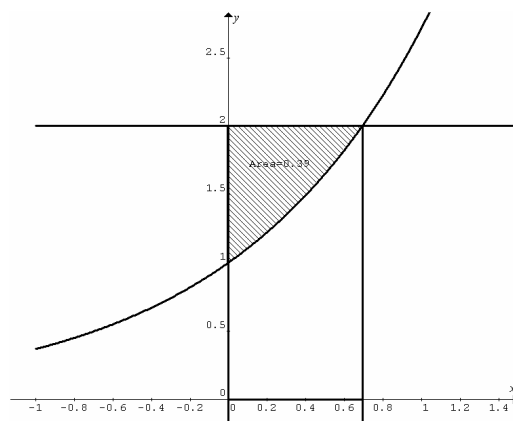
(c) $\Pr(-3 < x < 3) = \int_{-3}^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0.997$

(d) $\Pr(-\infty < x < \infty) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$.

Example 3 Find the area of the region bounded by the y -axis, the line $y = 2$ and the curve $y = e^x$.

For the curve $y = e^x$, when $y = 2$, $e^x = 2$, $\therefore x = \log_e 2$.

Area of the required region = area of the rectangular region – the area of the region under $y = e^x$ from $x = 0$ to $x = \log_e 2$
 $= 2 \log_e 2 - \int_0^{\log_e 2} e^x dx = 2 \log_e 2 - [e^x]_0^{\log_e 2} = 2 \log_e 2 - (2 - 1)$
 $= 0.39$

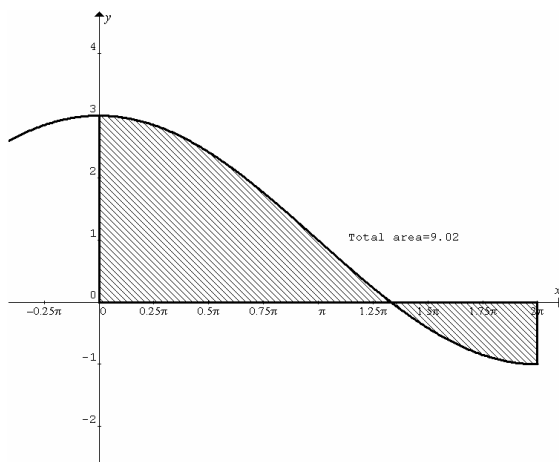


Example 4 Find the area of the region(s) bounded by the curve $y = 2 \cos \frac{x}{2} + 1$, the x -axis, the lines $x = 0$ and $x = 2\pi$.

Sketch the graph first. The curve crosses the x -axis. Find this x -intercept. Let $y = 0$, $2 \cos \frac{x}{2} + 1 = 0$, $\cos \frac{x}{2} = -\frac{1}{2}$, $\therefore \frac{x}{2} = \frac{2\pi}{3}$,

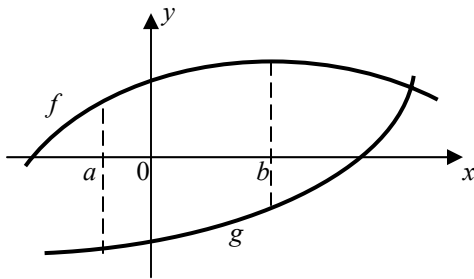
$\therefore x = \frac{4\pi}{3}$.

$$\begin{aligned} \text{Area} &= \left| \int_0^{\frac{4\pi}{3}} \left(2 \cos \frac{x}{2} + 1 \right) dx \right| + \left| \int_{\frac{4\pi}{3}}^{2\pi} \left(2 \cos \frac{x}{2} + 1 \right) dx \right| \\ &= \left[4 \sin \frac{x}{2} + x \right]_0^{\frac{4\pi}{3}} + \left[4 \sin \frac{x}{2} + x \right]_{\frac{4\pi}{3}}^{2\pi} \\ &= \left| 4 \sin \frac{2\pi}{3} + \frac{4\pi}{3} \right| + \left| \left(4 \sin \pi + 2\pi \right) - \left(4 \sin \frac{2\pi}{3} + \frac{4\pi}{3} \right) \right| \\ &= 2\sqrt{3} + \frac{4\pi}{3} + \left| 2\pi - 2\sqrt{3} - \frac{4\pi}{3} \right| \\ &= 2\sqrt{3} + \frac{4\pi}{3} + \left| \frac{2\pi}{3} - 2\sqrt{3} \right| \\ &= 2\sqrt{3} + \frac{4\pi}{3} + 2\sqrt{3} - \frac{2\pi}{3} = 4\sqrt{3} + \frac{2\pi}{3}. \end{aligned}$$



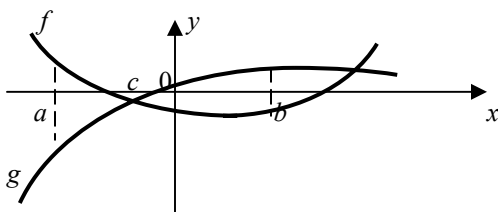
Area of region(s) between two curves

If $f(x) \geq g(x)$ in the interval $[a, b]$, or $g(x) \geq f(x)$ in the interval $[a, b]$, the area of the region between the two curves is given by $area = \left| \int_a^b (f(x) - g(x)) dx \right|$.



If the two curves cross at $x = c$ in the interval $[a, b]$, the area of the regions between the two curves is given by

$$area = \left| \int_a^c (f(x) - g(x)) dx \right| + \left| \int_c^b (f(x) - g(x)) dx \right|$$



The x -coordinates of the intersections are found by solving $y = f(x)$ and $y = g(x)$ simultaneously.

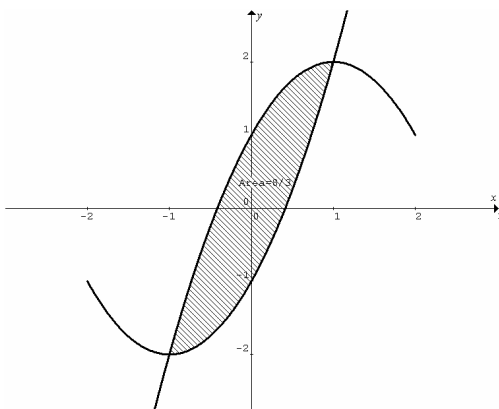
Example 1 Find the area of the region enclosed by $y = x^2 + 2x - 1$ and $y = -x^2 + 2x + 1$.

Solve the two equations simultaneously to find the x -coordinates of the intersections.

$$x^2 + 2x - 1 = -x^2 + 2x + 1, \therefore 2x^2 = 2, \therefore x = \pm 1.$$

$$Area = \left| \int_{-1}^1 ((x^2 + 2x - 1) - (-x^2 + 2x + 1)) dx \right|$$

$$= \left| \int_{-1}^1 (2x^2 - 2) dx \right| = \left| \left[\frac{2x^3}{3} - 2x \right]_{-1}^1 \right| = \left| \left(\frac{2}{3} - 2 \right) - \left(-\frac{2}{3} + 2 \right) \right| = \frac{8}{3}.$$



Example 2 Find the area of the region(s) bounded by $y = \sin x$, $y = \frac{2}{\pi}x$, $x = 0$ and $x = \pi$.

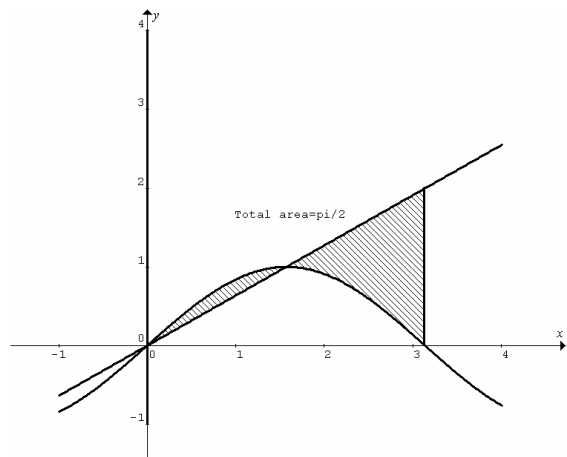
Solve $y = \sin x$ and $y = \frac{2}{\pi}x$ simultaneously to find $x = 0$ and $x = \frac{\pi}{2}$ in the interval $[0, \pi]$. Total area of bounded regions

$$= \left| \int_0^{\frac{\pi}{2}} \left(\sin x - \frac{2}{\pi}x \right) dx \right| + \left| \int_{\frac{\pi}{2}}^{\pi} \left(\sin x - \frac{2}{\pi}x \right) dx \right|$$

$$= \left| \left[-\cos x - \frac{x^2}{\pi} \right]_0^{\frac{\pi}{2}} \right| + \left| \left[-\cos x - \frac{x^2}{\pi} \right]_{\frac{\pi}{2}}^{\pi} \right|$$

$$= \left| \left(0 - \frac{\pi}{4} \right) - (-1 - 0) \right| + \left| (1 - \pi) - \left(0 - \frac{\pi}{4} \right) \right|$$

$$= -\frac{\pi}{4} + 1 + \frac{3\pi}{4} - 1 = \frac{\pi}{2}.$$



Note: The area of the enclosed regions is $\frac{1}{4}$ of the area of the rectangular region

