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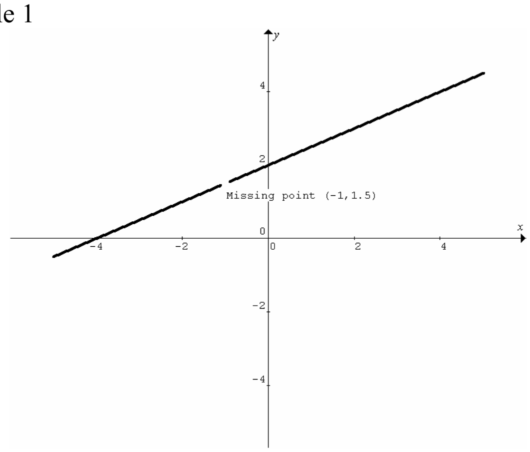
Calculus

Limits and continuity

Sometimes the value of a function $f(x)$ at $x = a$ may not be defined, but $f(x)$ may get close to certain value L as x gets close to a from both sides of a. We say L is the limit of $f(x)$ as x approaches a. This idea is expressed by the notation,
$\lim_{x \to a} f(x) = L$ or $\lim_{h \to 0} f(a+h) = L$, where $h = \Delta x$.

If x approaches a from the left, $x \to a^-$, h is a negative value.
If x approaches a from the right, $x \to a^+$, h is a positive value.

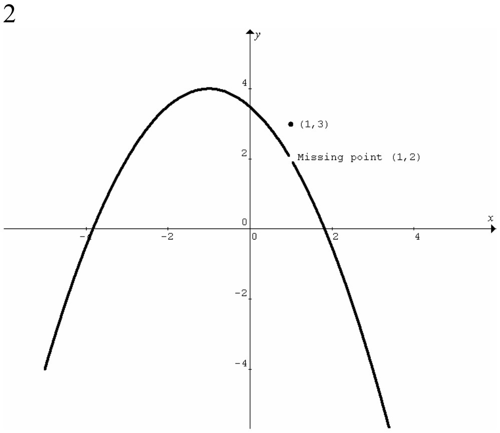
Example 1



The above function $f(x)$ is undefined at $x = -1$, i.e. $f(-1)$ does not exist. However, as $x \to -1$ from either side of $x = -1$, $f(x) \to 1.5$. $\therefore \lim_{x \to -1} f(x) = 1.5$ or $\lim_{h \to 0} f(-1+h) = 1.5$, i.e. the limit of $f(x)$ exists as x approaches -1 and it is 1.5.

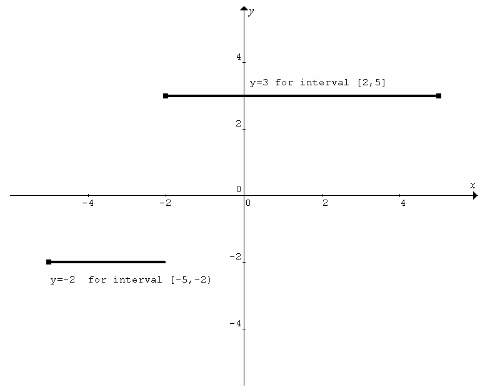
The function is **discontinuous** at $x = -1$.

Example 2



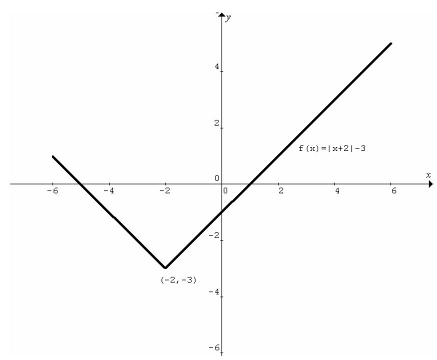
The function $f(x)$ is defined at $x = 1$, and $f(1) = 3$. However, as $x \to 1$ from either side of $x = 1$, $f(x) \to 2$. $\therefore \lim_{x \to 1} f(x) = 2$ or $\lim_{h \to 0} f(1+h) = 2$, i.e. the limit of $f(x)$ exists as x approaches 1 and it is 2. The function is discontinuous at $x = 1$.

Example 3



The above function $f(x)$ is defined at $x = -2$, and $f(-2) = 3$. As $x \to -2$ from the left side of $x = -2$, $f(x) \to -2$, and as $x \to -2$ from the right side, $f(x) \to 3$. The left and right limits are different, $\therefore \lim_{x \to -2} f(x)$ or $\lim_{h \to 0} f(-2+h)$ does not exist, and the function is discontinuous at $x = -2$.

Example 4



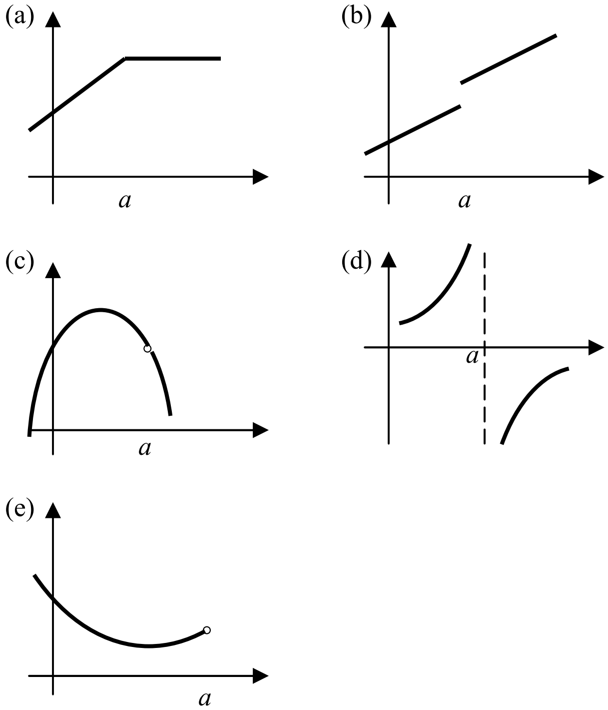
The limit of $f(x)$ exists as $x \to -2$, i.e. $\lim_{x \to -2} f(x) = -3$. The value of the function at $x = -2$ is defined, i.e. $f(-2) = -3$.
Since $\lim_{x \to -2} f(x) = f(-2)$, $\therefore f(x)$ is **continuous** at $x = -2$.
In general, a function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x)$ and $f(a)$ both exist and $\lim_{x \to a} f(x) = f(a)$.

Differentiability of a function at a point on an interval

A function $f(x)$ is differentiable at $x = a$ if it is continuous and there is no abrupt change in its gradient at $x = a$, i.e. the section on the left of $x = a$ is *smoothly* joined to the section on the right, and the curve appears to be a straight section (**local linearity**) in the immediate neighbourhood of $x = a$.

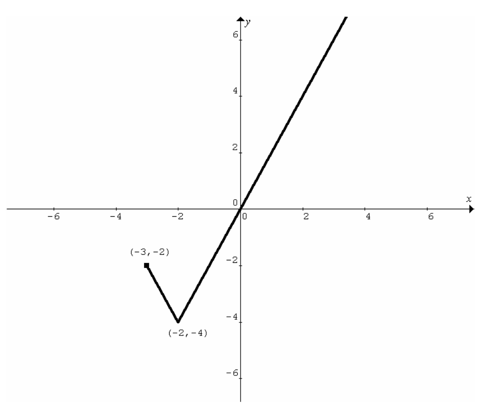
A function $f(x)$ is differentiable on a closed interval $[p, q]$ if $f(x)$ is differentiable at each point of the open interval (p, q) AND $\lim_{x \to p^+} \frac{f(x) - f(p)}{x - p}$ and $\lim_{x \to q^-} \frac{f(x) - f(q)}{x - q}$ both exist.
A function is **not** differentiable at an open end point.

Example 1 Discuss the differentiability of the following functions at $x = a$



(a) The function has an abrupt change in its gradient at $x = a$, $\therefore$ it is not differentiable at $x = a$.
(b) The function is not continuous at $x = a$, $\therefore$ it is not differentiable at $x = a$.
(c) The function is not continuous at $x = a$, $\therefore$ it is not differentiable at $x = a$.
(d) The function is undefined at $x = a$, $\therefore$ it is not differentiable at $x = a$.
(e) It is an open end point of the function at $x = a$, $\therefore$ it is not differentiable at $x = a$.

Example 2 Discuss the differentiability of the following absolute function $f(x)$ on the unbounded interval $[-3, \infty)$.



$f(x)$ is not differentiable at the vertex. $\therefore$ $f(x)$ is differentiable on the interval $[-3, -2)$ or $(-2, \infty)$.

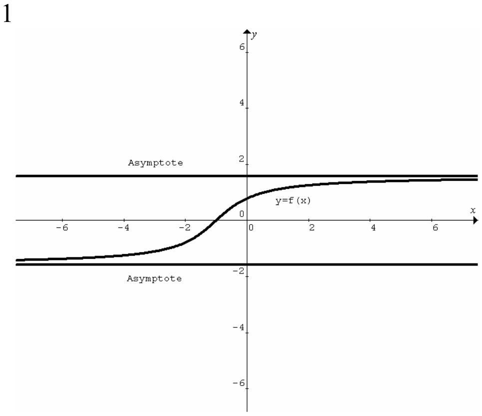
Graph of $f'(x)$ from graph of $f(x)$

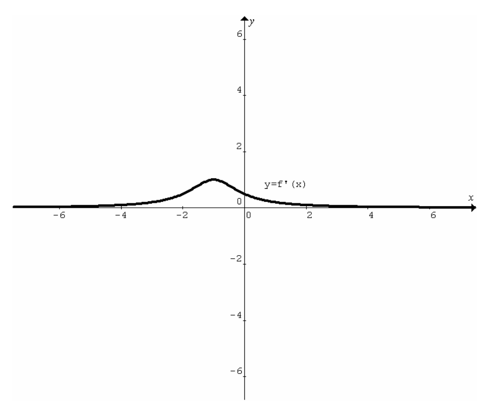
$f'(x)$ denotes the gradient function of function $f(x)$. $f'(x)$ is undefined at points where $f(x)$ is not differentiable.

Steps to follow in sketching the graph of $f'(x)$ from the graph of $f(x)$:
(1) Look for stationary points where $f'(x) = 0$. They are the x-intercepts of $f'(x)$ graph.
(2) Check the gradient (+/–) on each side of a stationary point of $f(x)$. The $f'(x)$ graph is above the x-axis for +, below the x-axis for −. If the two sides have opposite signs, the $f'(x)$ graph cuts across the x-axis. If the two sides have the same sign, the x-intercept of $f'(x)$ graph is a turning point.
(3) $f'(x)$ has the same vertical asymptotes as $f(x)$.
(4) The horizontal asymptote(s) of $f(x)$ corresponds to the asymptote $y = 0$ (i.e. the x-axis) of $f'(x)$.
(5) An oblique asymptote of $f(x)$ corresponds to a horizontal asymptote of $f'(x)$.

The graph of $f'(x)$ remains the same if $f(x)$ is vertically translated.

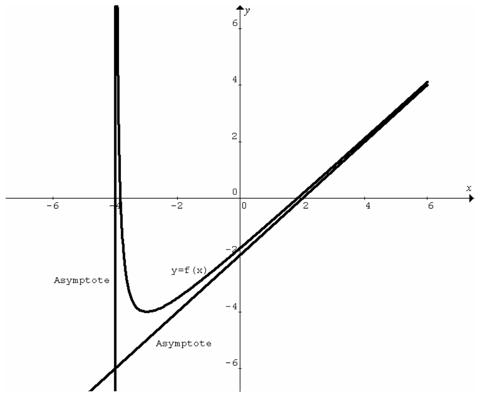
Example 1

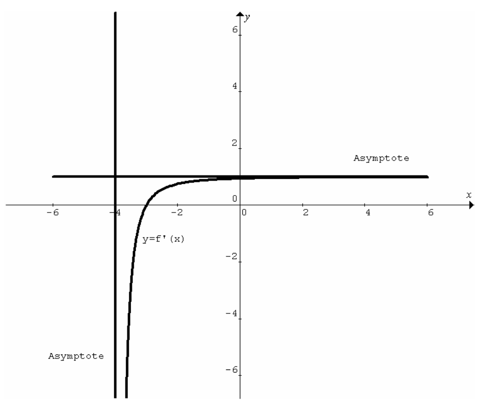




Domain of $f(x)$: *R*, domain of $f'(x)$: *R*.

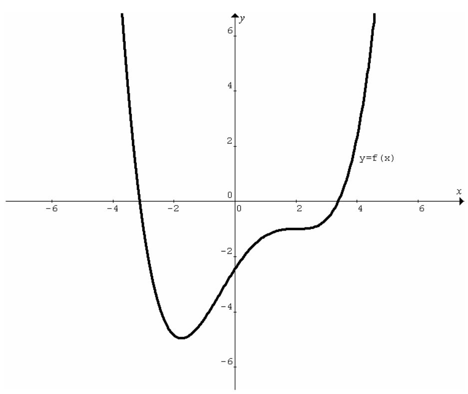
Example 2

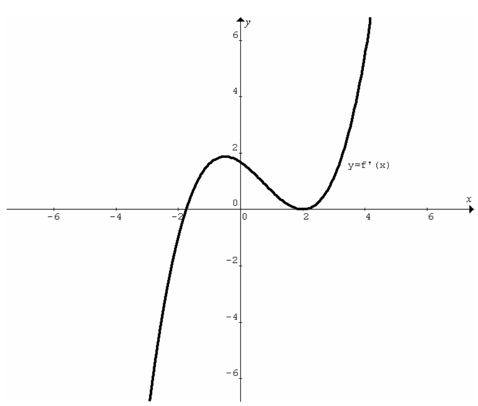




Domain of $f(x)$: $(-4,\infty)$, domain of $f'(x)$: $(-4,\infty)$.

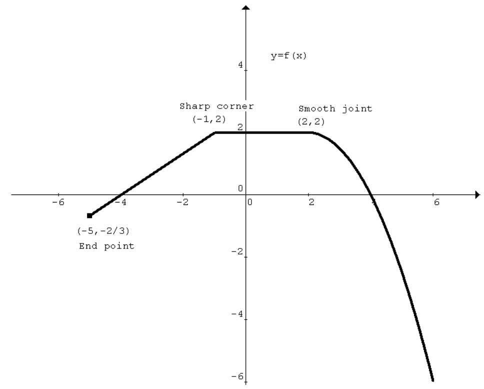
Example 3

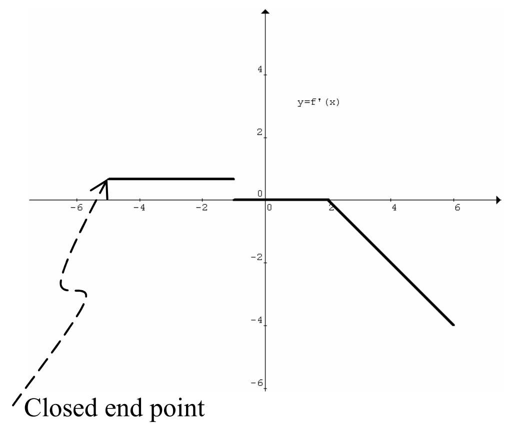




Domain of $f(x)$: R, domain of $f'(x)$: R.

Example 4





Domain of $f(x)$: $[-5,\infty)$,
domain of $f'(x)$: $[-5,-1)\cup(-1,\infty)$.

Rules for derivatives of x^n for $n \in Q$

The general rules are:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1},$$

$$\frac{d}{dx}\left(ax^n\right) = nax^{n-1},$$

$$\frac{d}{dx}\left((x-b)^n\right) = n(x-b)^{n-1},$$

$$\frac{d}{dx}\left(a(kx-b)^n\right) = kna(kx-b)^{n-1}.$$

Good to remember the derivative of $x^{\frac{1}{2}}$, ($\sqrt{x}$):

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}},$$

$$\frac{d}{dx}\left(a\sqrt{x}\right) = \frac{a}{2\sqrt{x}},$$

$$\frac{d}{dx}\left(\sqrt{ax-b}\right) = \frac{a}{2\sqrt{ax-b}}.$$

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Example 1 Find the derivative of $\dfrac{2\left(x+\frac{1}{2}\right)^{\frac{3}{2}}}{3}$.

$$\frac{d}{dx}\left(\frac{2\left(x+\frac{1}{2}\right)^{\frac{3}{2}}}{3}\right)=\frac{3}{2}\times\frac{2\left(x+\frac{1}{2}\right)^{\frac{1}{2}}}{3}=\left(x+\frac{1}{2}\right)^{\frac{1}{2}}.$$

Example 2 Given $y=2\sqrt{3x-5}$, find $\dfrac{dy}{dx}$ at $x=3$.

$$\frac{dy}{dx}=\frac{2\times 3}{2\sqrt{3x-5}}=\frac{3}{\sqrt{3x-5}}=\frac{3}{\sqrt{3(3)-5}}=\frac{3}{2}.$$

Example 3 Find the gradient function of $f(x)=\dfrac{5}{\sqrt{x}}-\dfrac{1}{2x+3}$.

$$f(x)=\frac{5}{\sqrt{x}}-\frac{1}{2x+3}=5x^{-\frac{1}{2}}-(2x+3)^{-1}.$$

$$f'(x)=-\frac{1}{2}\times 5x^{-\frac{1}{2}-1}-{}^{-}1\times 2(2x+3)^{-1-1}=-\frac{5}{2}x^{-\frac{3}{2}}+2(2x+3)^{-2}$$

or $-\dfrac{5}{2x^{\frac{3}{2}}}+\dfrac{2}{(2x+3)^2}$.

Example 4 Differentiate $\dfrac{3x^2+5x-1}{x^2}$.

$$\frac{3x^2+5x-1}{x^2}=\frac{3x^2}{x^2}+\frac{5x}{x^2}-\frac{1}{x^2}=3+5x^{-1}-x^{-2}.$$

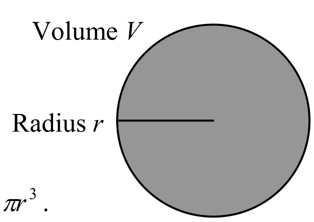
$$\frac{d}{dx}\left(3+5x^{-1}-x^{-2}\right)=0+{}^{-}1\times 5x^{-2}-{}^{-}2x^{-3}=-5x^{-2}+2x^{-3}$$

$$=-\frac{5}{x^2}+\frac{2}{x^3}\text{ or }\frac{-5x+2}{x^3}.$$

Example 5 Find the gradient of the tangent to the curve $y=\dfrac{5(1-3x)^2}{2}$ at $x=\dfrac{4}{15}$.

Gradient $=\dfrac{dy}{dx}={}^{-}3\times 2\times\dfrac{5(1-3x)}{2}={}^{-}15(1-3x)=-3$.

Example 6 Find the rate of change of the volume of a sphere with respect to its radius when the radius is 2 units.



Volume of a sphere $V(r)=\dfrac{4}{3}\pi r^3$.

Required rate $=\dfrac{dV}{dr}=4\pi r^2=4\pi(2)^2=16\pi$ sq. units.

Rules for derivatives of e^x and $\log_e x$

The general rules are: $\dfrac{d}{dx}\left(e^x\right)=e^x$,

$$\frac{d}{dx}\left(ae^{kx}\right)=kae^{x},$$

$$\frac{d}{dx}\left(e^{x-b}\right)=e^{x-b},$$

$$\frac{d}{dx}\left(ae^{kx-b}\right)=kae^{kx-b}.$$

$$\frac{d}{dx}\left(\log_e x\right)=\frac{1}{x},$$

$$\frac{d}{dx}\left(a\log_e(kx)\right)=\frac{a}{x},$$

$$\frac{d}{dx}\left(\log_e(x-b)\right)=\frac{1}{x-b},$$

$$\frac{d}{dx}\left(a\log_e(kx-b)\right)=\frac{ka}{kx-b},$$

$$\frac{d}{dx}\left(a\log_e k(x-b)\right)=\frac{a}{x-b}.$$

Note: The above rules are for exponential and logarithmic functions with base *e* only. See examples below for other bases.

Example 1 Find the derivative of $3e^{1-2x}$.

$$\frac{d}{dx}\left(3e^{1-2x}\right)={}^{-}2\times 3e^{1-2x}=-6e^{1-2x}.$$

Example 2 Find the gradient of $f(x)=1-2\log_e 3(x+1)$ at $x=0$.

$f'(x)=-\dfrac{2}{x+1}$. Gradient $=f'(0)=-2$ at $x=0$.

Example 3 Given $x=\dfrac{e^{2t+1}-e^{1-2t}}{e^{1-t}}$, find the rate of change of x with respect to t.

$$x=\frac{e^{2t+1}-e^{1-2t}}{e^{1-t}}=e^{2t+1-(1-t)}-e^{1-2t-(1-t)}=e^{3t}-e^{-t}.$$

$$\frac{dx}{dt}=3e^{3t}+e^{-t}.$$

Example 4 Differentiate $\log_e\dfrac{(2x+1)(x-1)}{x^2-4}$.

$$\log_e\frac{(2x+1)(x-1)}{x^2-4}=\log_e\frac{(2x+1)(x-1)}{(x-2)(x+2)}$$

$$=\log_e(2x+1)+\log_e(x-1)-\log_e(x-2)-\log_e(x+2).$$

$$\frac{d}{dx}\left(\log_e\frac{(2x+1)(x-1)}{x^2-4}\right)=\frac{2}{2x+1}+\frac{1}{x-1}-\frac{1}{x-2}-\frac{1}{x+2}.$$

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Example 5 Find the gradient function of $f(x)=3\log_{10}(x+1)$.

Change to base *e*: $f(x)=\dfrac{3\log_e(x+1)}{\log_e 10}$.

$f'(x)=\dfrac{3}{(x+1)\log_e 10}$.

Example 6 Find the gradient of the tangent to the curve $y=a\times 2^{\frac{x}{3}}$ at $x=3$ in terms of *a*.

Change to base *e*: $y=a\times e^{\log_e 2^{\frac{x}{3}}}=ae^{\frac{x}{3}\log_e 2}=ae^{\frac{\log_e 2}{3}x}$.

$$\frac{dy}{dx}=\frac{\log_e 2}{3}\left(ae^{\frac{\log_e 2}{3}x}\right)=\frac{\log_e 2}{3}\left(a\times 2^{\frac{x}{3}}\right).$$

At $x=3$, gradient $=\dfrac{2a\log_e 2}{3}$.

Rules for the derivatives of $\sin(x)$**,** $\cos(x)$**,** $\tan(x)$

The general rules are: $\frac{d}{dx}(\sin(x))=\cos(x)$,

$\frac{d}{dx}(a\sin(kx))=ka\cos(kx)$,

$\frac{d}{dx}(\sin(x-b))=\cos(x-b)$,

$\frac{d}{dx}(a\sin(kx-b))=ka\cos(kx-b)$,

$\frac{d}{dx}(a\sin k(x-b))=ka\cos k(x-b)$.

$\frac{d}{dx}(\cos(x))=-\sin(x)$,

$\frac{d}{dx}(a\cos(kx))=-ka\sin(kx)$,

$\frac{d}{dx}(\cos(x-b))=-\sin(x-b)$,

$\frac{d}{dx}(a\cos(kx-b))=-ka\sin(kx-b)$,

$\frac{d}{dx}(a\cos k(x-b))=-ka\sin k(x-b)$.

$\frac{d}{dx}(\tan(x))=\sec^2(x)$, $\left[\sec(x)=\dfrac{1}{\cos(x)}\right]$,

$\frac{d}{dx}(a\tan(kx))=ka\sec^2(kx)$,

$\frac{d}{dx}(\tan(x-b))=\sec^2(x-b)$,

$\frac{d}{dx}(a\tan(kx-b))=ka\sec^2(kx-b)$,

$\frac{d}{dx}(a\tan k(x-b))=ka\sec^2 k(x-b)$.

Example 1 Given $y=-\dfrac{2}{3}\cos 3\pi(x+5)$, find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}={}^-3\pi\times\frac{^-2}{3}\sin 3\pi(x+5)=2\pi\sin 3\pi(x+5).$$

Example 2 Find $\dfrac{d}{dx}\left(\dfrac{3\sin\left(2x-\frac{\pi}{3}\right)}{5}\right)$.

$$\frac{d}{dx}\left(\frac{3\sin\left(2x-\frac{\pi}{3}\right)}{5}\right)=\frac{2\times 3\cos\left(2x-\frac{\pi}{3}\right)}{5}=\frac{6}{5}\cos\left(2x-\frac{\pi}{3}\right).$$

Example 3 Find the derivative of $y=\dfrac{\sin\frac{\pi}{6}(x+1)}{2\cos\frac{\pi}{6}(x+1)}$.

Change to $y=\dfrac{1}{2}\tan\left(\dfrac{\pi}{6}(x+1)\right)$.

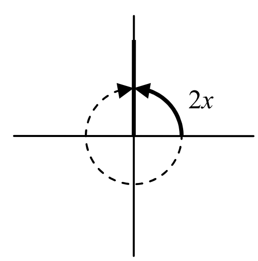
$$y'=\frac{\pi}{6}\times\frac{1}{2}\sec^2\left(\frac{\pi}{6}(x+1)\right)=\frac{\pi}{12}\sec^2\left(\frac{\pi}{6}(x+1)\right).$$

Example 4 Find the derivative of
$y=10^{10}\sin^2(\pi x)+10^{10}\cos^2(\pi x)+10^{10}$.

Simplify, $y=10^{10}\sin^2(\pi x)+10^{10}\cos^2(\pi x)+10^{10}$
$=10^{10}\left(\sin^2(\pi x)+\cos^2(\pi x)+1\right)$
$=10^{10}(1+1)=2\times 10^{10}$. It is a constant. Hence, $y'=0$.

Example 5 Find $-\pi\le x\le\pi$ where the gradient of the curve $y=3\cos(2x)-1$ is –6.

Gradient $=\dfrac{dy}{dx}=2\times{}^-3\sin(2x)=-6\sin(2x)$. Let $\dfrac{dy}{dx}=-6$.
$\therefore -6\sin(2x)=-6$, $\sin(2x)=1$.
Solve $\sin(2x)=1$ for *x*, where $-\pi\le x\le\pi$, i.e. $-2\pi\le 2x\le 2\pi$.



$2x=-\dfrac{3\pi}{2}$ or $\dfrac{\pi}{2}$, $\therefore x=-\dfrac{3\pi}{4}$ or $\dfrac{\pi}{4}$.

Derivatives of linear combinations of functions

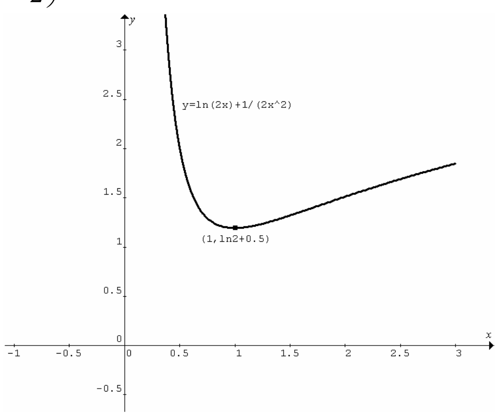
Example 1 Find the stationary point(s) of $y = \log_e(2x) + \frac{1}{2x^2}$.

The implicit domain of the function is R^+.

At the stationary points, $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^3} = 0$, $\therefore \frac{x^2 - 1}{x^3} = 0$, $\therefore x^2 - 1 = 0$ and $x > 0$.

Hence $x = 1$ and $\therefore y = \log_e 2 + \frac{1}{2}$. The stationary point is $\left(1, \log_e 2 + \frac{1}{2}\right)$.

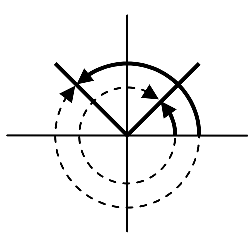


Example 2 Find the x-coordinates of the stationary points of $y = 2x + \sqrt{2}\cos(2x)$, where $-\pi < x < \pi$.

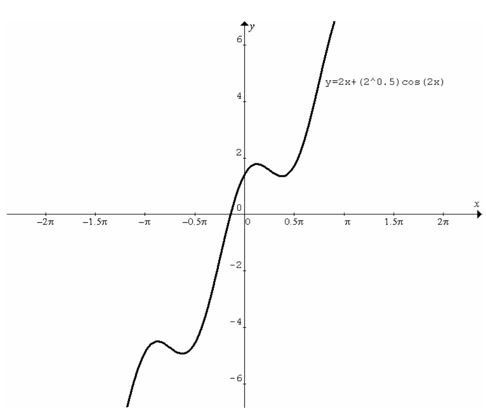
$\frac{dy}{dx} = 2 - 2\sqrt{2}\sin(2x)$. Let $\frac{dy}{dx} = 0$.

$\therefore 2 - 2\sqrt{2}\sin(2x) = 0$ and $-2\pi < 2x < 2\pi$.

$\therefore \sin(2x) = \frac{1}{\sqrt{2}}$.



$\therefore 2x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$. $\therefore x = -\frac{7\pi}{8}, -\frac{5\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}$.



Example 3 Find the stationary point of $y = 2e^x + 3e^{-x} - 5x$.

$\frac{dy}{dx} = 2e^x - 3e^{-x} - 5 = 0$. Multiply by e^x.

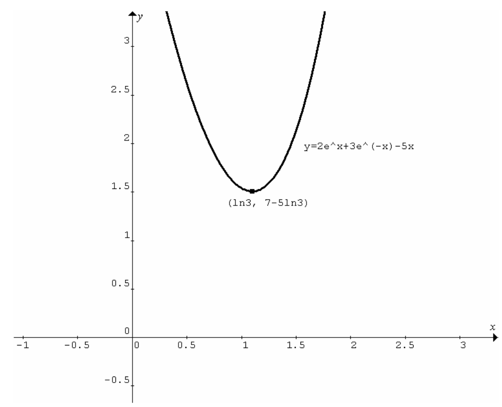
$2(e^x)^2 - 5(e^x) - 3 = 0$. Factorise.

$(2e^x + 1)(e^x - 3) = 0$.

Since $2e^x + 1 \neq 0$, $\therefore e^x - 3 = 0$, $\therefore e^x = 3$ or $x = \log_e 3$.

$\therefore y = 2(3) + 3\left(\frac{1}{3}\right) - 5\log_e 3 = 7 - 5\log_e 3$

The stationary point is $(\log_e 3, 7 - 5\log_e 3)$.



The chain rule

The chain rule is used to differentiate *composite*[*] functions.
([*] See 'Functions and graphs')

Given $y = f(u(x))$, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = f'(u) \times u'(x)$.

Example 1 Find the derivative of $\sqrt{4 - x^2}$.

Let $u = 4 - x^2$ and $y = \sqrt{u}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times (-2x) = \frac{-x}{\sqrt{4 - x^2}}$.

Example 2 Given $f(x) = \sin(x^2 + 1)$, find $f'(x)$.

Let $u = x^2 + 1$ and $\therefore f(u) = \sin(u)$.

$f'(x) = f'(u) \times u'(x) = \cos(u) \times 2x = 2x\cos(x^2 + 1)$.

Example 3 Differentiate $e^{\cos(x) + \sin(x)}$ with respect to *x*.

Let $u = \cos(x) + \sin(x)$ and $y = e^u$.

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times (-\sin(x) + \cos(x))$

$= (\cos(x) - \sin(x))e^{\cos(x) + \sin(x)}$.

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Example 4 Find the derivative of $\log_e \sqrt{x+1}$.

It can be done without applying the chain rule.

Re-express: $\log_e \sqrt{x+1} = \log_e (x+1)^{\frac{1}{2}} = \frac{1}{2}\log_e(x+1)$, where $x+1>0$, i.e. $x>-1$.

$\frac{d}{dx}\left(\log_e \sqrt{x+1}\right) = \frac{d}{dx}\left(\frac{1}{2}\log_e(x+1)\right) = \frac{1}{2(x+1)}$.

Using the chain rule:

Let $u=\sqrt{x+1}$,

$\frac{d}{dx}\left(\log_e \sqrt{x+1}\right) = \frac{d}{dx}\left(\log_e u\right)\times\frac{du}{dx} = \frac{1}{u}\times\frac{1}{2\sqrt{x+1}} = \frac{1}{2(x+1)}$.

Example 5 Given $y=e^{2\log_e\left(x^2+1\right)}$, find $\frac{dy}{dx}$.

Re-express:

$y=e^{2\log_e\left(x^2+1\right)} = e^{\log_e\left(x^2+1\right)^2} = \left(x^2+1\right)^2 = x^4+2x^2+1$,

$\frac{dy}{dx} = 4x^3+4x^2 = 4x^2(x+1)$.

Using the chain rule will involve more steps to get the same result.

The product rule

Use the chain rule to differentiate products of functions.

If $y=u(x)v(x)$, then $\frac{dy}{dx} = v\frac{du}{dx}+u\frac{dv}{dx}$ or $y' = vu'+uv'$.

Example 1 Find $\frac{dy}{dx}$, given $y=(3x-2)^4(1-2x)^3$.

Rather than expanding the expression, use the product rule with $u=(3x-2)^4$ and $v=(1-2x)^3$.

$u' = 3\times4(3x-2)^3 = 12(3x-2)^3$,

$v' = {}^-2\times3(1-2x)^2 = -6(1-2x)^2$.

$\therefore y' = (1-2x)^3\times12(3x-2)^3 + (3x-2)^4\times{}^-6(1-2x)^2$

$= 12(1-2x)^3(3x-2)^3 - 6(3x-2)^4(1-2x)^2$

$= 6(1-2x)^2(3x-2)^3\left[2(1-2x)-(3x-2)\right]$

$= 6(1-2x)^2(3x-2)^3(4-7x)$

Example 2 Differentiate $(x-1)^3\sin(2x)$.

Let $u=(x-1)^3$, $v=\sin(2x)$, $\therefore y=uv$.

$u' = 3(x-1)^2$, $v' = 2\cos(2x)$.

$y' = \sin(2x)\times3(x-1)^2 + (x-1)^3\times2\cos(2x)$

$= (x-1)^2\left(3\sin(2x)+2(x-1)\cos(2x)\right)$

Example 3 Find the stationary point of $y=2xe^{2x}$.

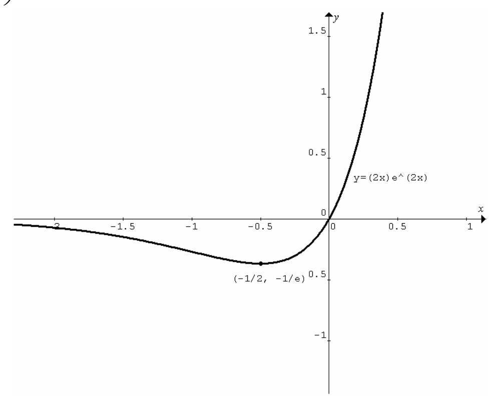
Let $u=2x$ and $v=e^{2x}$. $u'=2$ and $v'=2e^{2x}$.

$y' = e^{2x}\times2+2x\times2e^{2x} = 2e^{2x}(1+2x)$.

At stationary point, $y'=0$. $\therefore 2e^{2x}(1+2x)=0$.

Since $2e^{2x}\neq0$, $\therefore 1+2x=0$, $x=-\frac{1}{2}$ and

$y=2\left(-\frac{1}{2}\right)e^{2\left(-\frac{1}{2}\right)} = -e^{-1} = -\frac{1}{e}$. The stationary point is $\left(-\frac{1}{2},-\frac{1}{e}\right)$.



The quotient rule

For functions of the form $f(x)=\frac{u(x)}{v(x)}$, $f'(x)=\frac{vu'-uv'}{v^2}$. This is called the quotient rule. It can also be expressed as

$\frac{dy}{dx} = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$.

Example 1 Given $f(x)=\tan(2x)$, show that $f'(x)=2\sec^2(2x)$.

Re-express $\tan(2x)=\frac{\sin(2x)}{\cos(2x)}$. Let $u=\sin(2x)$, $v=\cos(2x)$ and $f(x)=\frac{u}{v}$. $u'=2\cos(2x)$, $v'=-2\sin(2x)$,

$f'(x)=\frac{vu'-uv'}{v^2} = \frac{\cos(2x)\times2\cos(2x)-\sin(2x)\times{}^-2\sin(2x)}{\cos^2(2x)}$

$= \frac{2\left(\cos^2(2x)+\sin^2(2x)\right)}{\cos^2(2x)} = \frac{2}{\cos^2(2x)} = 2\sec^2(2x)$.

Example 2 Find $\frac{d}{dx}\left(\frac{\log_e x}{x}\right)$.

$\frac{d}{dx}\left(\frac{\log_e x}{x}\right) = \frac{x\times\frac{d}{dx}\left(\log_e x\right)-\log_e x\times\frac{d}{dx}(x)}{x^2}$

$= \frac{x\times\frac{1}{x}-\log_e x\times1}{x^2} = \frac{1-\log_e x}{x^2}$.

Example 3 Given $y = \frac{x^2}{\sqrt{1+x}}$, find y'.

Let $u = x^2$ and $v = \sqrt{1+x}$. $u' = 2x$, $v' = \frac{1}{2\sqrt{1+x}}$.

$y' = \frac{vu' - uv'}{v^2} = \frac{\sqrt{1+x} \times 2x - x^2 \times \frac{1}{2\sqrt{1+x}}}{(1+x)}$. Multiply the numerator and denominator by $2\sqrt{1+x}$.

$$y' = \frac{4x(1+x) - x^2}{2(1+x)\sqrt{1+x}} = \frac{3x^2 + 4x}{2(1+x)(1+x)^{\frac{1}{2}}} = \frac{x(3x+4)}{2(1+x)^{\frac{3}{2}}}.$$

Example 4 Find the derivative of $x^{-2}e^x$.

Use either the product rule or the quotient rule.

To apply the quotient rule, re-express $x^{-2}e^x$ as $\frac{e^x}{x^2}$. Let $u = e^x$, $v = x^2$ and $y = \frac{u}{v}$. $u' = e^x$, $v' = 2x$ and

$$y' = \frac{vu' - uv'}{v^2} = \frac{x^2e^x - e^x \times 2x}{x^4} = \frac{xe^x(x-2)}{x^4} = \frac{e^x(x-2)}{x^3}.$$

Anti-differentiation

Standard notation for anti-differentiation of function $f(x)$ is $\int f(x)dx$. Anti-differentiation is the reverse process of differentiation. They undo each other.

$\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$ and $\int\left(\frac{d}{dx}(f(x))\right)dx = f(x) + C$, where C is a constant. The inclusion of C in the second expression is due to the fact that $\frac{d}{dx}(f(x)+C) = \frac{d}{dx}(f(x))$.

$f(x)+C$	$\frac{d}{dx}(f(x)+C)$
$x^2 - 1000000$	$2x$
x^2	$2x$
$x^2 + 1000000$	$2x$

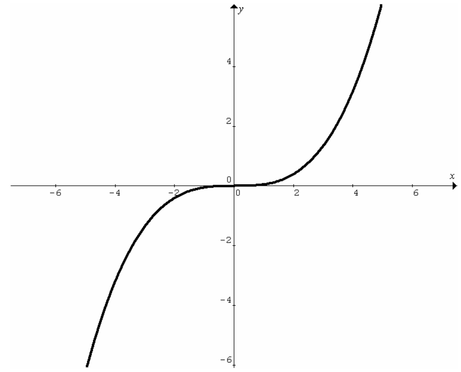
Relation between graph of anti-derivative function and graph of original function

Since $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$, $\therefore$ the gradient of the graph of the anti-derivative function gives the value of the original function. This is the same relationship as the graph of $f(x)$ and the graph of $f'(x)$ discussed earlier (page 2).

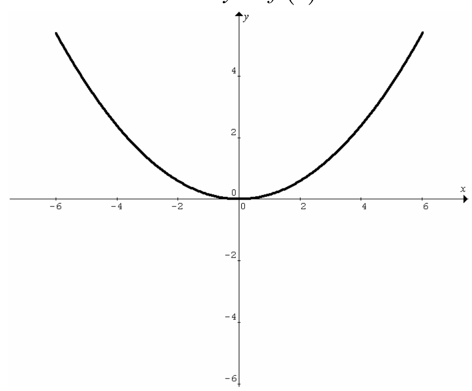
There are infinite number of anti-derivative graphs that correspond to the graph of the original function.

Example 1

$y = \int f(x)dx$

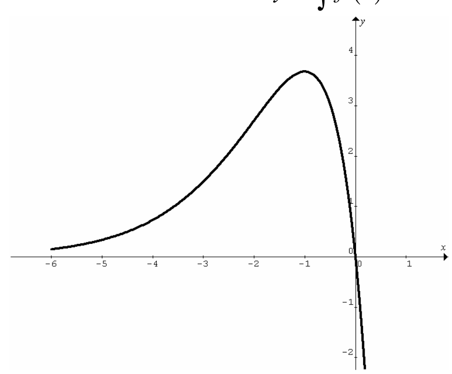


$y = f(x)$

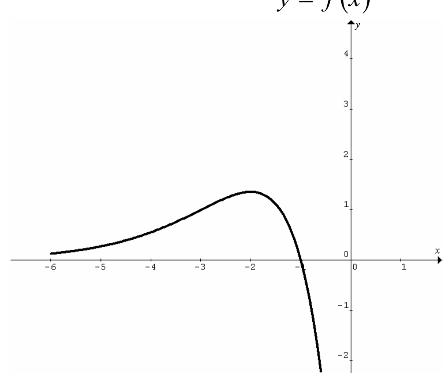


Example 2

$y = \int f(x)dx$



$y = f(x)$



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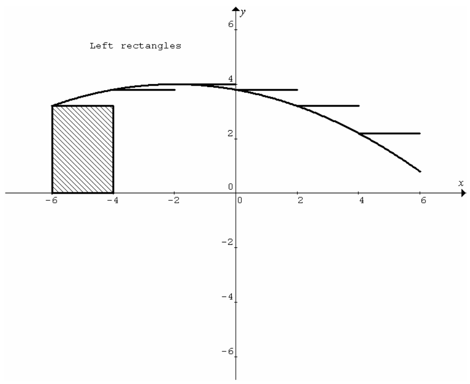
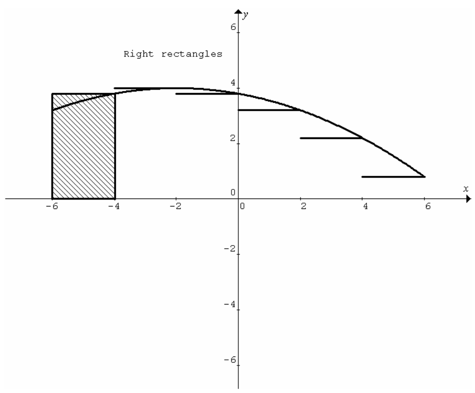
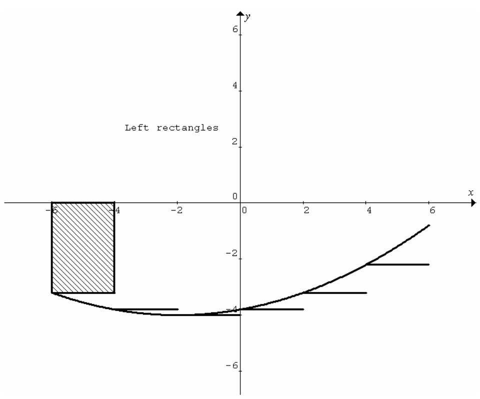
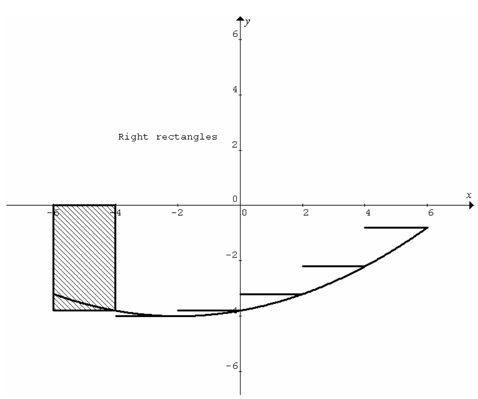
Approximate area ‘under’ a curve by left rectangles and right rectangles









Example 1 Estimate the area bounded by the curve $y=-0.05(x+2)^2+4$, the x-axis, $x=-6$ and $x=6$ by left rectangles 2 units wide. (See diagram on the left)

The first left rectangle has $y=f(-6)=3.2$ as its height, the height of the second left rectangle $f(-4)=3.8$, the third $f(-2)=4$, etc.

$Area \approx 3.2\times 2+3.8\times 2+4\times 2+3.8\times 2+3.2\times 2+2.2\times 2$

$=(3.2+3.8+4+3.8+3.2+2.2)\times 2=40.4$ square units.

Example 2 Estimate the area bounded by the curve $y=-0.05(x+2)^2+4$, the x-axis, $x=-6$ and $x=6$ by right rectangles 2 units wide. (See diagram on the left)

The first right rectangle has $y=f(-4)=3.8$ as its height, the height of the second right rectangle $f(-2)=4$, the third $f(0)=3.8$, etc.

$Area \approx 3.8\times 2+4\times 2+3.8\times 2+3.2\times 2+2.2\times 2+0.8\times 2$

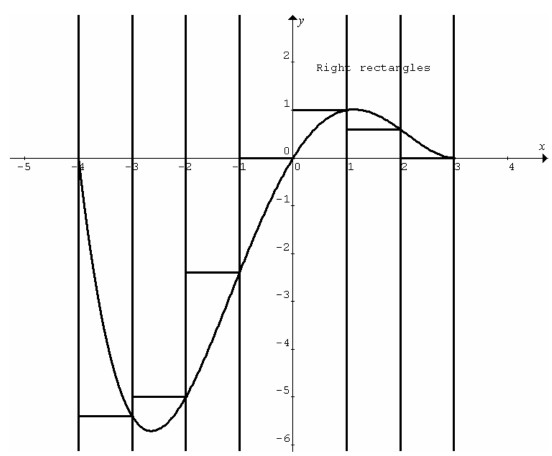
$=(3.8+4+3.8+3.2+2.2+0.8)\times 2=35.6$ square units.

Note: The average of the above two results give a better estimate of the required area, i.e. $\frac{40.2+35.6}{2}=37.9$ square units.

Using graphics calculator, evaluate $\int f(x)dx=38.4$ square units.

Example 3 Use right rectangles 1 unit wide to estimate the area bounded by the curve $y=0.1x(x+4)(x-3)^2$ and the x-axis.

The given function is in factorised form, the linear factors indicate the location of the x-intercepts: $x=-4,0,3$. It is a turning point at $x=3$.



For $-4\le x\le 0$, $f(x)\le 0$, $\therefore$ the height of a rectangle $=|f(a)|$ for $-4\le a\le 0$.

x	- 3	- 2	- 1	0	1	2	3
$\lvert f(x)\rvert$	5.4	5	2.4	0	1	0.6	0

$Area \approx (5.4+5+2.4+0+1+0.6+0)\times 1=14.4$ square units.

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The fundamental theorem of calculus

If F is any anti-derivative of f on interval $[a,b]$, i.e.

$F(x)=\int f(x)dx$, then $\int_a^b f(x)dx = F(b)-F(a)$.

This statement is known as the fundamental theorem of calculus.

$\int_a^b f(x)dx$ is called a **definite integral**, and $\int f(x)dx$, an anti-derivative of $f(x)$ is also called an **indefinite** integral.

Example 1 Given the anti-derivative of $f(x)$ is $\sin x+\frac{x^2+1}{5}$, evaluate $\int_0^{\frac{\pi}{2}} f(x)dx$.

$$\int_0^{\frac{\pi}{2}} f(x)dx=\left(\sin\frac{\pi}{2}+\frac{\pi^2+4}{20}\right)-\left(\sin(0)+\frac{1}{5}\right)=1+\frac{\pi^2+4}{20}-\frac{1}{5}$$

$$=\frac{\pi^2}{20}+1.$$

Example 2 Given the anti-derivative of $g(x)$ is $\frac{\log_e(x+1)}{x+1}$, evaluate $\int_0^1 g(x)dx$.

$$\int_0^1 g(x)dx=\frac{\log_e 2}{2}-\frac{\log_e 1}{1}=\frac{1}{2}\log_e 2 \text{ or } \log_e\sqrt{2}.$$

In practice an intermediate step is added before the substitutions of a and b. $\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)$.

Example 3 The anti-derivative of $h(x)$ is $\sin^2 x-\frac{1}{\cos^2 x}$, evaluate $\int_0^t h(x)dx$, where $0<t<\frac{\pi}{2}$.

$$\int_0^t h(x)dx=\left[\sin^2 x-\frac{1}{\cos^2 x}\right]_0^t=\left(\sin^2 t-\frac{1}{\cos^2 t}\right)-\left(0-\frac{1}{\cos^2 0}\right)$$

$$=\sin^2 t-\frac{1}{\cos^2 t}+1.$$

Properties of anti-derivatives and definite integrals

$$\int Af(x)dx=A\int f(x)dx$$

$$\int (f(x)\pm g(x))dx=\int f(x)dx\pm\int g(x)dx$$

$$\int_a^b f(x)dx+\int_b^c f(x)dx=\int_a^c f(x)dx$$

$$\int_a^b f(x)dx=-\int_b^a f(x)dx$$

$$\int_a^a f(x)dx=0$$

Rules for anti-derivatives of x^n, where $n\in Q$

For $n\neq -1$,

$$\int x^n dx=\frac{x^{n+1}}{n+1}+C$$

$$\int ax^n dx=\frac{ax^{n+1}}{n+1}+C$$

$$\int (x-b)^n dx=\frac{(x-b)^{n+1}}{n+1}+C$$

$$\int a(kx-b)^n dx=\frac{a(kx-b)^{n+1}}{k(n+1)}+C.$$

For $n=-1$, i.e. x^{-1} or $\frac{1}{x}$, where $x\neq 0$,

$$\int \frac{1}{x}dx=\log_e|x|+C$$

$$\int \frac{a}{kx}dx=\frac{a}{k}\log_e|x|+C$$

$$\int \frac{1}{x-b}dx=\log_e|x-b|+C$$

$$\int \frac{a}{k(x-b)}dx=\frac{a}{k}\log_e|x-b|+C$$

$$\int \frac{a}{kx-b}dx=\frac{a}{k}\log_e|kx-b|+C.$$

Rules for anti-derivative of e^{kx}

$$\int e^x dx=e^x+C$$

$$\int ae^{kx}dx=\frac{a}{k}e^{kx}+C$$

$$\int e^{x-b}dx=e^{x-b}+C$$

$$\int ae^{k(x-b)}dx=\frac{a}{k}e^{k(x-b)}+C$$

$$\int ae^{kx-b}dx=\frac{a}{k}e^{kx-b}+C$$

Rules for anti-derivatives of $\cos(kx)$ and $\sin(kx)$

$$\int \cos(x)dx=\sin(x)+C$$

$$\int a\cos(kx)dx=\frac{a}{k}\sin(kx)+C$$

$$\int \cos(x-b)dx=\sin(x-b)+C$$

$$\int a\cos k(x-b)dx=\frac{a}{k}\sin k(x-b)+C$$

$$\int a\cos(kx-b)dx=\frac{a}{k}\sin(kx-b)+C$$

$$\int \sin(x)dx=-\cos(x)+C$$

$$\int a\sin(kx)dx=-\frac{a}{k}\cos(kx)+C$$

$$\int \sin(x-b)dx=-\cos(x-b)+C$$

$$\int a\sin k(x-b)dx=-\frac{a}{k}\cos k(x-b)+C$$

$$\int a\sin(kx-b)dx=-\frac{a}{k}\cos(kx-b)+C$$

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Example 1 Find ***an*** anti-derivative of $\frac{-3}{5(2x+1)^2}$.

$$\int \frac{-3}{5(2x+1)^2}dx = \int \frac{-3(2x+1)^{-2}}{5}dx = \frac{-3(2x+1)^{-1}}{5(-1)(2)} + C = \frac{3}{10(2x+1)} + C$$

Choose any real value for C, usually 0 for convenience.

Example 2 Find $F(x) = \int \frac{2}{3x-4}dx$ such that $F(1) = 1$.

$F(x) = \frac{2}{3}\log_e|3x-4| + C$, $F(1) = \frac{2}{3}\log_e|-1| + C = 1$,

$\therefore \frac{2}{3}\log_e(1) + C = 1$, $C = 1$. $\therefore F(x) = \frac{2}{3}\log_e|3x-4| + 1$.

Example 3 Given $f'(x) = \cos\pi(x+1) - 2e^{\pi x}$, find $f(x)$ such that $f(0) = -\frac{2}{\pi}$.

$f'(x) = \cos\pi(x+1) - 2e^{\pi x}$, $\therefore f(x) = \int(\cos\pi(x+1) - 2e^{\pi x})dx$

$= \frac{1}{\pi}\sin\pi(x+1) - \frac{2}{\pi}e^{\pi x} + C$.

$f(0) = \frac{1}{\pi}\sin\pi - \frac{2}{\pi}e^0 + C = -\frac{2}{\pi}$, $\therefore C = 0$.

$\therefore f(x) = \frac{1}{\pi}\sin\pi(x+1) - \frac{2}{\pi}e^{\pi x}$.

Example 4 Evaluate $\int_{-1}^{1}(x^2-1)(x^2+1)dx$.

$$\int_{-1}^{1}(x^2-1)(x^2+1)dx = \int_{-1}^{1}(x^4-1)dx = \left[\frac{x^5}{5} - x\right]_{-1}^{1}$$

$$= \left(\frac{1}{5} - 1\right) - \left(\frac{-1}{5} + 1\right) = -\frac{8}{5}.$$

Example 5 Evaluate $\int_0^{\log_e 2}\frac{e^{2x} - e^{-2x}}{e^{2x}}dx$.

$$\int_0^{\log_e 2}\frac{e^{2x} - e^{-2x}}{e^{2x}}dx = \int_0^{\log_e 2}(1 - e^{-4x})dx = \left[x + \frac{1}{4}e^{-4x}\right]_0^{\log_e 2}$$

$$= \left(\log_e 2 + \frac{1}{4}e^{-4\log_e 2}\right) - \left(0 + \frac{1}{4}e^0\right) = \log_e 2 + \frac{1}{4}\times\frac{1}{16} - \frac{1}{4}$$

$$= \log_e 2 - \frac{15}{64}.$$

Example 6 Evaluate $\int_0^1 \frac{x^2+1}{x+1}dx$.

$$\int_0^1 \frac{x^2+1}{x+1}dx = \int_0^1\left(x - 1 + \frac{2}{x+1}\right)dx = \left[\frac{x^2}{2} - x + 2\log_e(x+1)\right]_0^1$$

$$= \left(\frac{1}{2} - 1 + 2\log_e 2\right) - (0) = 2\log_e 2 - \frac{1}{2}.$$

Integration by recognition

Example 1 Find $\int \frac{1}{\cos^2 x}dx$.

$\int \frac{1}{\cos^2(x)}dx = \int \sec^2(x)dx = \tan(x) + C$ by recognising that $\frac{d}{dx}(\tan(x)) = \sec^2(x)$.

Example 2 Find the derivative of $x\log_e x$. Hence find the anti-derivative of $\log_e x$.

Let $y = x\log_e x$, apply the product rule to obtain

$\frac{dy}{dx} = (\log_e x)(1) + (x)\left(\frac{1}{x}\right) = \log_e x + 1$,

$\therefore \log_e x = \frac{dy}{dx} - 1$. $\int \log_e x dx = \int\left(\frac{dy}{dx} - 1\right)dx = \int\frac{dy}{dx}dx - \int 1dx$

$= y - x + C = x\log_e x - x + C$.

Example 3 Find $f'(x)$, given $f(x) = \sin(x^{-1})$. Hence evaluate $\int_{-\frac{2}{\pi}}^{\frac{2}{\pi}}\frac{\cos(x^{-1})}{x^2}dx$.

$f(x) = \sin(x^{-1})$, use the chain rule to obtain

$f'(x) = -x^{-2}\cos(x^{-1})$.

$$\therefore \int_{-\frac{2}{\pi}}^{\frac{2}{\pi}}\frac{\cos(x^{-1})}{x^2}dx = \int_{-\frac{2}{\pi}}^{\frac{2}{\pi}} - f'(x)dx = [-f(x)]_{-\frac{2}{\pi}}^{\frac{2}{\pi}} = [-\sin(x^{-1})]_{-\frac{2}{\pi}}^{\frac{2}{\pi}}$$

$$= \left(-\sin\left(\frac{\pi}{2}\right)\right) - \left(-\sin\left(-\frac{\pi}{2}\right)\right) = -1 - 1 = -2.$$

Example 4 Given $y = xe^{-2x}$, find $\frac{dy}{dx}$ and hence evaluate $\int_0^1 xe^{-2x}dx$.

$y = xe^{-2x}$, $\therefore \frac{dy}{dx} = (x)(-2e^{-2x}) + (1)(e^{-2x}) = -2xe^{-2x} + e^{-2x}$

$\therefore xe^{-2x} = \frac{1}{2}\left(e^{-2x} - \frac{dy}{dx}\right)$, $\therefore \int_0^1 xe^{-2x}dx = \int_0^1 \frac{1}{2}\left(e^{-2x} - \frac{dy}{dx}\right)dx$

$$= \left[\frac{1}{2}\left(\frac{e^{-2x}}{-2} - xe^{-2x}\right)\right]_0^1 = \frac{1}{2}\left(\frac{e^{-2}}{-2} - e^{-2}\right) - \frac{1}{2}\left(\frac{1}{-2} - 0\right)$$

$$= \frac{1}{4} - \frac{3e^{-2}}{4} = \frac{1}{4}(1 - 3e^{-2}).$$

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