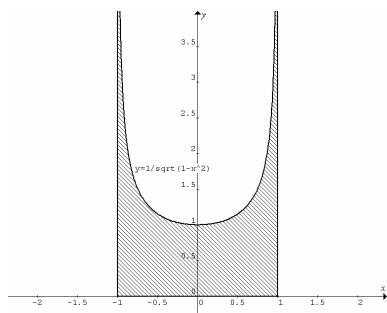


Further Calculus Part II

Applications of integration

Areas of regions bounded by curves

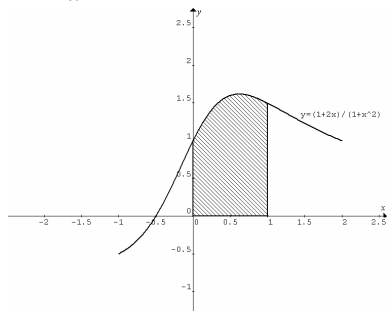
Example 1 Find the exact area of the region bounded by the curve $y = \frac{1}{\sqrt{1-x^2}}$, $x = -1$, $x = 1$ and the x -axis.



$$Area = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_{-1}^1 = \sin^{-1}(1) - \sin^{-1}(-1) = \pi.$$

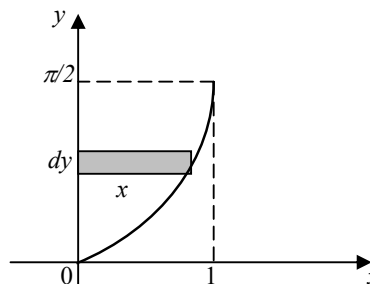
Note: Although $\frac{1}{\sqrt{1-x^2}}$ is undefined at $x = \pm 1$, the definite integral $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx$ is finite.

Example 2 Find the exact area of the region bounded by the curve $y = \frac{1+2x}{1+x^2}$ and the x -axis from $x = 0$ to $x = 1$.



$$\begin{aligned} Area &= \int_0^1 \frac{1+2x}{1+x^2} dx = \int_0^1 \left(\frac{1}{1+x^2} + \frac{2x}{1+x^2} \right) dx \\ &= \int_0^1 \frac{1}{1+x^2} dx + \int_0^1 \frac{2x}{1+x^2} dx \\ &= [\tan^{-1} x]_0^1 + \int_0^1 \frac{1}{u} du = [\tan^{-1} x]_0^1 + \int_1^2 \frac{1}{u} du \\ &= [\tan^{-1} x]_0^1 + [\log_e |u|]_1^2 = \frac{\pi}{4} + \log_e 2. \end{aligned}$$

Example 3 (a) Find the exact area of the region bounded by the curve $y = \sin^{-1} x$ and the y -axis from $y = 0$ to $y = \frac{\pi}{2}$.
(b) Hence find the exact area of the region bounded by the curve $y = \sin^{-1} x$ and the x -axis from $x = 0$ to $x = 1$.



(a) $y = \sin^{-1} x$, $\therefore x = \sin y$.

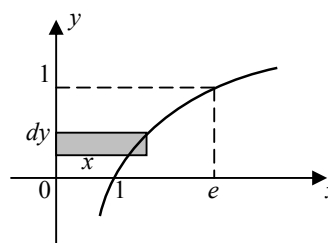
$$\begin{aligned} Area &= \int_0^{\pi/2} x dy = \int_0^{\pi/2} \sin y dy = [-\cos y]_0^{\pi/2} \\ &= \left(-\cos \frac{\pi}{2} \right) - (-\cos 0) = 1. \end{aligned}$$

(b) The required area is obtained by subtracting the answer in (a) from the area of the rectangular region $1 \times \frac{\pi}{2}$.

$\therefore Area = \frac{\pi}{2} - 1$. Note: direct evaluation of $\int_0^1 \sin^{-1} x dx$ is not required for exam purpose. However, students are expected to find $\int_0^1 \sin^{-1} x dx \approx 0.5708$ by graphics calculator.

Example 4 Find the exact value of $\int_1^e \log_e x dx$.

$\int_1^e \log_e x dx$ gives the area of the region between the curve $y = \log_e x$ and the x -axis from $x = 1$ to $x = e$. Direct evaluation of $\int_1^e \log_e x dx$ is not required.



When $x = e$, $y = 1$.

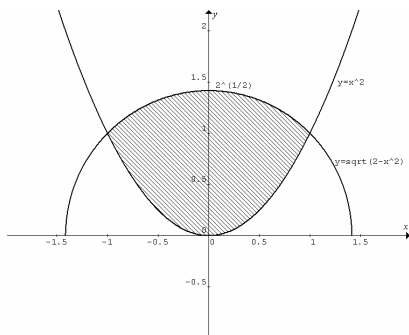
Area of the region bounded by the curve $y = \log_e x$ and the y -axis from $y = 0$ to $y = 1$:

$$\int_0^1 x dy = \int_0^1 e^y dy = [e^y]_0^1 = e - 1.$$

Area of the region bounded by the curve $y = \log_e x$ and the x -axis from $x = 1$ to $x = e$ is obtained by subtracting the area $(e-1)$ from the area of the rectangular region $e \times 1$.

$$\therefore \int_1^e \log_e x dx = e \times 1 - (e-1) = 1.$$

Example 5 (a) Use graphics calculator to find the area of the region bounded by the curves $y = x^2$ and $y = \sqrt{2-x^2}$.
 (b) Hence find the area of the regions bounded by the two curves and the x -axis.



(a) Solve $y = x^2$ and $y = \sqrt{2-x^2}$ simultaneously to find the x -coordinates of the intersections.

$$x^2 = \sqrt{2-x^2}, \quad (x^2)^2 + x^2 - 2 = 0, \quad \therefore (x^2 + 2)(x^2 - 1) = 0, \\ \therefore x = \pm 1.$$

Area of region bounded by the two curves:

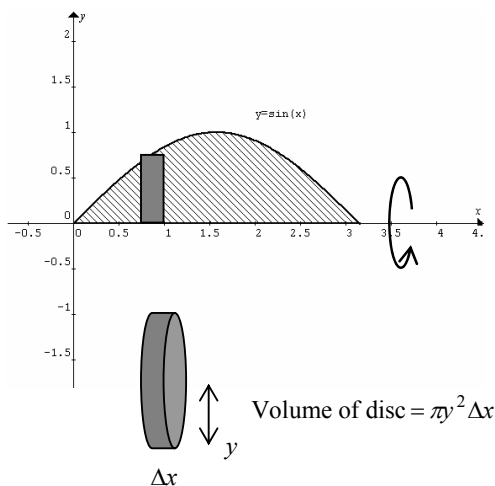
$$\int_{-1}^1 (\sqrt{2-x^2} - x^2) dx \approx 1.904 \quad (\text{Graphics calculator})$$

(b) Area of the regions bounded by the two curves and the x -axis is given by subtracting answer (a) from the area of the semi-circular region:

$$\text{Area} \approx \frac{1}{2} (\pi(\sqrt{2})^2) - 1.904 = 1.238$$

Volumes of solids of revolution of a region about the x -axis, and the y -axis

Example 1 The region bounded by the curve $y = \sin x$, the x -axis, $x = 0$ and $x = \pi$ is rotated about the x -axis. Calculate the volume of the solid of revolution.



$$\text{Volume of the solid of revolution} = \lim_{\Delta x \rightarrow 0} \sum \pi y^2 \Delta x$$

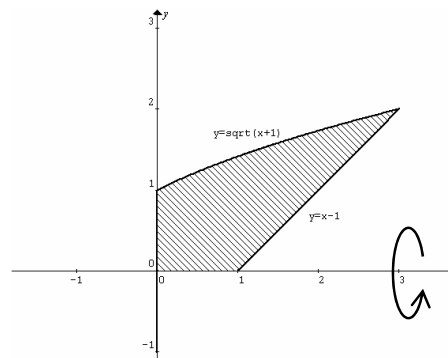
$$= \int_0^\pi \pi y^2 dx = \int_0^\pi \pi \sin^2 x dx = \int_0^\pi \frac{\pi}{2} (1 - \cos 2x) dx$$

$$= \left[\frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right) \right]_0^\pi = \frac{\pi^2}{2}.$$

Example 2 Find the exact volume of the solid generated by rotating about the x -axis the region bounded by the curve $y = \sqrt{x+1}$, the line $y = x-1$, the x -axis, and the y -axis.

Coordinates of the intersection: $x-1 = \sqrt{x+1}$,

$(x-1)^2 = x+1$, $x^2 - 3x = 0$, $x(x-3) = 0$, only $x = 3$ satisfies the equation, $y = 2$.



Volume of the solid = volume generated by $y = \sqrt{x+1}$ minus volume generated by $y = x-1$

$$= \int_0^3 \pi (\sqrt{x+1})^2 dx - \text{volume of cone}$$

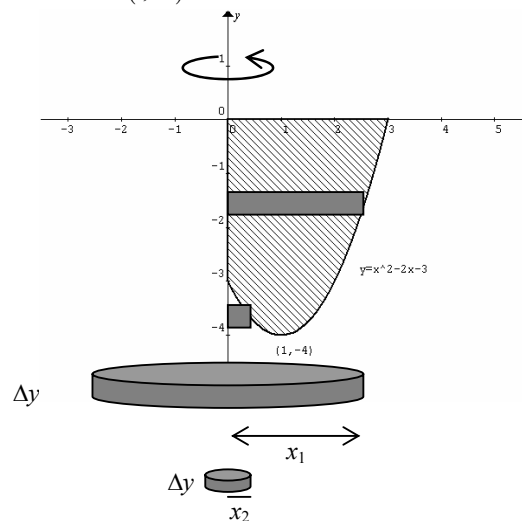
$$= \int_0^3 \pi (x+1) dx - \frac{1}{3} \pi (2)^2 (2)$$

$$= \left[\frac{\pi}{2} (x+1)^2 \right]_0^3 - \frac{8\pi}{3} = 8\pi - \frac{\pi}{2} - \frac{8\pi}{3} = \frac{29\pi}{6}.$$

Example 3 The region bounded by $y = x^2 - 2x - 3$, the x -axis, $x = 0$ and $x = 3$ is rotated about the y -axis. Calculate the volume of the solid of revolution.

$$y = x^2 - 2x - 3 = (x-1)^2 - 4.$$

The curve cuts the y -axis at $y = -3$, and the minimum point is found to be $(1, -4)$.



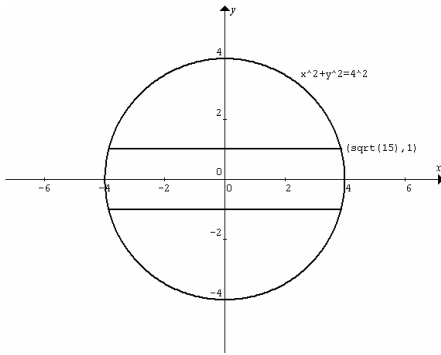
$$\text{Since } y = x^2 - 2x - 3 = (x-1)^2 - 4,$$

$$\therefore x_1 = \sqrt{y+4} + 1 \text{ or } x_2 = -\sqrt{y+4} + 1$$

$$\begin{aligned} \therefore \text{Volume} &= \int_{-4}^0 \pi(x_1)^2 dy - \int_{-4}^{-3} \pi(x_2)^2 dy \\ &= \int_{-4}^0 \pi(y+5+2\sqrt{y+4})^2 dy - \int_{-4}^{-3} \pi(y+5-2\sqrt{y+4})^2 dy \\ &= \left[\pi \left(\frac{1}{2}y^2 + 5y + \frac{4}{3}(y+4)^{\frac{3}{2}} \right) \right]_{-4}^0 - \left[\pi \left(\frac{1}{2}y^2 + 5y - \frac{4}{3}(y+4)^{\frac{3}{2}} \right) \right]_{-4}^{-3} \\ &= \pi \left(\frac{32}{3} \right) - \pi(-12) - \pi \left(-\frac{71}{6} \right) + \pi(-12) = \frac{45\pi}{2}. \end{aligned}$$

Example 4 A hole of radius 1.0 cm is drilled through the centre of a sphere of radius 4.0 cm. Find the volume removed.

The volume removed consists of the volumes of the cylinder (radius 1.0 cm, length $2\sqrt{15}$ cm) and two end caps.



$$\begin{aligned} \text{Volume of each end cap} &= \int_{\sqrt{15}}^4 \pi y^2 dx = \int_{\sqrt{15}}^4 \pi(16 - x^2) dx \\ &= \left[\pi \left(16x - \frac{1}{3}x^3 \right) \right]_{\sqrt{15}}^4 = \pi \left(\frac{128}{3} - 11\sqrt{15} \right) = 0.2006 \end{aligned}$$

$$\begin{aligned} \therefore \text{Total volume removed} \\ &= \pi(1.0)^2(2\sqrt{15}) + 2 \times 0.2006 = 24.7359 \text{ cm}^3. \end{aligned}$$

Differential equations

An equation involving derivatives is a differential equation.

$\frac{dy}{dx} = f(x)$, $\frac{d^2y}{dx^2} = f(x)$, $\frac{dy}{dx} = f(y)$ etc. are examples of differential equations.

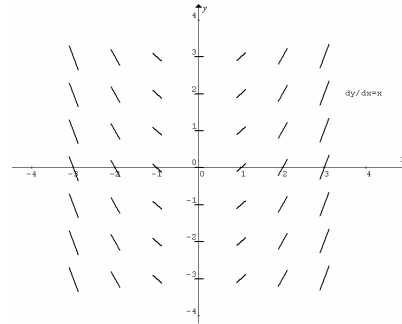
To solve a differential equation is to find the relationship between the two variables x and y . Usually y is expressed in terms of x .

Direction (slope) field of a differential equation

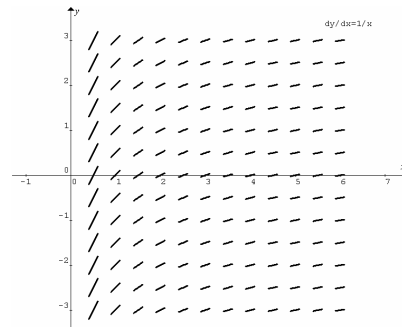
Given a differential equation, a 'map' showing the slope of the relation or function can be drawn up. At each point (x, y) in the coordinate plane, a short line segment with a

slope given by $\frac{dy}{dx} = f(x)$, $f(y)$ or $f(x, y)$ is marked for that point. The resulting 'map' is called a direction (slope) field of the differential equation.

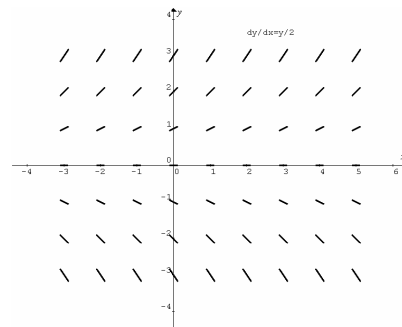
Example 1 Draw a direction field of $\frac{dy}{dx} = x$ for $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.



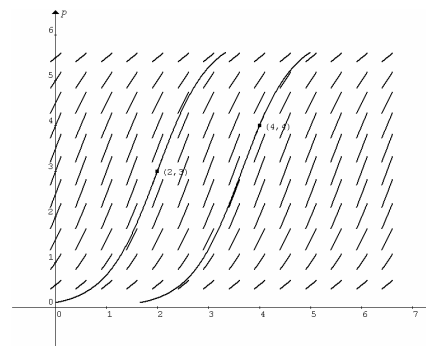
Example 2 Draw a direction field of $\frac{dy}{dx} = \frac{1}{x}$ for $0 < x \leq 6$ and $-3 \leq y \leq 3$.



Example 3 Draw a slope field of $\frac{dy}{dx} = \frac{y}{2}$ for $-3 \leq x \leq 5$ and $-3 \leq y \leq 3$.



Example 4 Sketch two particular solution curves over the following direction field. One satisfies the initial condition $P(2) = 3$, the other satisfies $P(4) = 4$.



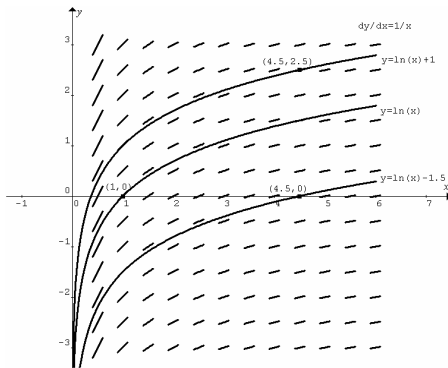
A slope field enables one to find the general solution to a differential equation. If initial conditions are known, a particular solution can be determined.

The slope field in example 2 indicates that $y = \log_e x + C$ is the general solution to $\frac{dy}{dx} = \frac{1}{x}$. Three particular solutions shown below correspond to three different initial conditions represented by the three points $(4.5, 2.5)$, $(1, 0)$ and $(4.5, 0)$.

The top curve passes through $(4.5, 2.5)$, \therefore it is $y = \log_e x + 1$.

The middle curve passes through $(1, 0)$, it is $y = \log_e x$.

The bottom curve passes through $(4.5, 0)$, it is $y = \log_e x - 1.5$.



Techniques in solving differential equations

(1) Type $\frac{dy}{dx} = f(x)$

Example 1 Solve $\frac{dy}{dx} = -e^{2x}$, given $y = 0$ when $x = 0$.

$$\frac{dy}{dx} = -e^{2x}, y = \int -e^{2x} dx = -\frac{1}{2}e^{2x} + C.$$

$$y = 0 \text{ when } x = 0, \therefore 0 = -\frac{1}{2}e^0 + C, \therefore C = \frac{1}{2}.$$

$$\therefore y = -\frac{1}{2}e^{2x} + \frac{1}{2}.$$

Note: $y = -\frac{1}{2}e^{2x} + C$ is called the *general solution*, and

$y = -\frac{1}{2}e^{2x} + \frac{1}{2}$ is a *particular solution* of the differential equation.

Example 2 Solve $2x \frac{dy}{dx} = 1$, given $y = 1$ when $x = e$.

$$2x \frac{dy}{dx} = 1, \frac{dy}{dx} = \frac{1}{2x}, y = \int \frac{1}{2x} dx = \frac{1}{2} \log_e |x| + C.$$

$$y = 1 \text{ when } x = e, \therefore 1 = \frac{1}{2} \log_e |e| + C, \therefore C = \frac{1}{2}.$$

$$\therefore y = \frac{1}{2} \log_e |x| + \frac{1}{2}.$$

Example 3 Verify that $y = 1 + \sqrt{1+x^2}$ is a solution to the equation $\sqrt{1+x^2} \frac{dy}{dx} - x = 0$.

$$y = 1 + \sqrt{1+x^2}, \text{ apply the chain rule, } \frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}.$$

$$\text{Substitute, } LHS = \sqrt{1+x^2} \left(\frac{x}{\sqrt{1+x^2}} \right) - x = 0 = RHS.$$

$\therefore y = 1 + \sqrt{1+x^2}$ is a solution because it satisfies the differential equation.

Example 4 Verify that $y = \log_e(x^2 + 2x - 3)$ is a solution to the equation $\left(\frac{x+1}{2} - \frac{2}{x+1} \right) \frac{dy}{dx} = 1$.

$$y = \log_e(x^2 + 2x - 3), \text{ apply the chain rule, } \frac{dy}{dx} = \frac{2x+2}{x^2+2x-3} = \frac{2(x+1)}{x^2+2x-3}.$$

$$\text{Substitute, } LHS = \left(\frac{x+1}{2} - \frac{2}{x+1} \right) \frac{2(x+1)}{x^2+2x-3}$$

$$= \left(\frac{(x+1)^2 - 4}{2(x+1)} \right) \frac{2(x+1)}{x^2+2x-3} = 1 = RHS.$$

$y = \log_e(x^2 + 2x - 3)$ satisfies the differential equation, \therefore it is a solution.

(2) Type $\frac{d^2y}{dx^2} = f(x)$

Example 1 Find the general solution to the equation

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{1-x}}.$$

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{1-x}},$$

$$\frac{dy}{dx} = \int \frac{1}{\sqrt{1-x}} dx = \int (1-x)^{-\frac{1}{2}} dx = -2(1-x)^{\frac{1}{2}} + C,$$

$$y = \int \left(-2(1-x)^{\frac{1}{2}} + C \right) dx = \frac{4}{3}(1-x)^{\frac{3}{2}} + Cx + D$$

Example 2 Find the particular solution to the differential

equation $\frac{d^2x}{dt^2} = -\cos 2t$, given $\frac{dx}{dt} = 0$ and $x = -1$ when

$t = 0$.

$$\frac{d^2x}{dt^2} = -\cos 2t,$$

$$\frac{dx}{dt} = \int -\cos 2t dt = -\frac{1}{2} \sin 2t + C,$$

$$\frac{dx}{dt} = 0 \text{ when } t = 0, \therefore C = 0 \text{ and } \frac{dx}{dt} = -\frac{1}{2} \sin 2t,$$

$$x = \int -\frac{1}{2} \sin 2t dt = \frac{1}{4} \cos 2t + D,$$

$$x = -1 \text{ when } t = 0, \therefore -1 = \frac{1}{4} + D, \therefore D = -\frac{5}{4}.$$

$$\text{Hence } x = \frac{1}{4} \cos 2t - \frac{5}{4}.$$

Example 3 Verify that $y = 3e^{-2x}$ is a solution to the

differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$.

$$y = 3e^{-2x}, \quad \frac{dy}{dx} = -6e^{-2x}, \quad \frac{d^2y}{dx^2} = 12e^{-2x}.$$

Substitute, $LHS = 12e^{-2x} + 3(-6e^{-2x}) + 2(3e^{-2x}) = 0 = RHS$,

$\therefore y = 3e^{-2x}$ is a solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$.

(3) Type $\frac{dy}{dx} = f(y)$

It can be shown that $\frac{dx}{dy} = 1/\frac{dy}{dx}$.

Therefore, the differential equation $\frac{dy}{dx} = f(y)$ can be changed to $\frac{dx}{dy} = \frac{1}{f(y)}$.

Example 1 Solve $\frac{dy}{dx} = -2y$ for y in terms of x .

$$\frac{dy}{dx} = -2y, \quad \therefore \frac{dx}{dy} = -\frac{1}{2y}, \quad \therefore x = \int -\frac{1}{2y} dy,$$

$$\therefore x = -\frac{1}{2} \log_e |y| + c.$$

Express y in terms of x :

$$|y| = e^{-2(x-c)} = e^{2c} e^{-2x} = Ae^{-2x}, \quad \text{where } A = e^{2c} > 0.$$

Hence $y = \pm Ae^{-2x}$ for $x \in R$.

$$\text{Verification: } y = Ae^{-2x}, \quad \frac{dy}{dx} = -2Ae^{-2x} = -2y.$$

$$y = -Ae^{-2x}, \quad \frac{dy}{dx} = 2Ae^{-2x} = -2y.$$

Example 2 Given $y = 0$ when $x = 0$, find the particular

solution of $\frac{dy}{dx} = \frac{1}{1-y}$.

$$\frac{dy}{dx} = \frac{1}{1-y},$$

$$\therefore \frac{dx}{dy} = 1-y, \quad \therefore x = \int (1-y) dy = -\frac{1}{2}(1-y)^2 + C.$$

$$y = 0 \text{ when } x = 0, \quad \therefore C = \frac{1}{2} \text{ and } \therefore x = -\frac{1}{2}(1-y)^2 + \frac{1}{2}.$$

Express y in terms of x :

$$2x = -(1-y)^2 + 1, \quad (1-y)^2 = 1-2x, \quad 1-y = \pm\sqrt{1-2x},$$

$$y = 1 \pm \sqrt{1-2x}.$$

Note: $y = 1 - \sqrt{1-2x}$ is the only solution that satisfies the condition $y = 0$ when $x = 0$.

Example 3 Find the general solution to the differential

equation $\frac{dy}{dx} = \frac{1}{4}(y^2 - 4)$.

$$\frac{dy}{dx} = \frac{1}{4}(y^2 - 4),$$

$$\frac{dx}{dy} = \frac{4}{y^2 - 4} = \frac{4}{(y-2)(y+2)} = \frac{1}{y-2} - \frac{1}{y+2},$$

$$x = \int \left(\frac{1}{y-2} - \frac{1}{y+2} \right) dy = \log_e |y-2| - \log_e |y+2| + c$$

$$\therefore x = \log_e \left| \frac{y-2}{y+2} \right| + c = \log_e \left| \frac{y-2}{y+2} \right| + c.$$

Hence $\left| \frac{y-2}{y+2} \right| = e^{x-c} = e^x e^{-c} = Ae^x$, where $A = e^{-c} > 0$.

$$\therefore \frac{y-2}{y+2} = \pm Ae^x, \quad y-2 = \pm Ae^x(y+2),$$

$$y-2 = \pm Ae^x y \pm 2Ae^x, \quad \therefore y \mp Ae^x y = 2 \pm 2Ae^x,$$

$$(1 \mp Ae^x)y = 2(1 \pm Ae^x), \quad \therefore y = \frac{2(1 \pm Ae^x)}{1 \mp Ae^x}.$$

Example 4 Find x when $y = 1$, given $y = 0$ when $x = 0$ and

$$\frac{dy}{dx} = \frac{1}{(y-1)(y-2)}.$$

$$\frac{dy}{dx} = \frac{1}{(y-1)(y-2)}, \quad \therefore \frac{dx}{dy} = (y-1)(y-2) = y^2 - 3y + 2,$$

$$x = \int (y^2 - 3y + 2) dy = \frac{1}{3}y^3 - \frac{3}{2}y^2 + 2y + C.$$

$y = 0$ when $x = 0$, $\therefore C = 0$.

$$\text{When } y = 1, \quad x = \frac{1}{3} - \frac{3}{2} + 2 = \frac{5}{6}.$$

Example 5 Find the general solution to the differential

equation $\frac{dy}{dx} = \sqrt{9-y^2}$.

$$\frac{dy}{dx} = \sqrt{9-y^2}, \quad \frac{dx}{dy} = \frac{1}{\sqrt{9-y^2}}, \quad x = \int \frac{1}{\sqrt{9-y^2}} dy,$$

$$\therefore x = \text{Sin}^{-1} \left(\frac{y}{3} \right) + c. \quad \text{Hence } y = 3 \sin(x-c).$$

Example 6 Find the particular solution to the differential

equation $\frac{dy}{dx} = -\sqrt{1-2y^2}$, which satisfies the initial

condition when $x = 0, y = 0$.

$$\frac{dy}{dx} = -\sqrt{1-2y^2}, \quad \frac{dx}{dy} = -\frac{1}{\sqrt{1-(\sqrt{2}y)^2}},$$

$$x = \int -\frac{1}{\sqrt{1-(\sqrt{2}y)^2}} dy. \quad \therefore x = -\frac{1}{\sqrt{2}} \text{Cos}^{-1}(\sqrt{2}y) + C.$$

$$\text{When } x = 0, y = 0, \quad \therefore 0 = \frac{1}{\sqrt{2}} \left(\frac{\pi}{2} \right) + C, \quad \therefore C = -\frac{\pi}{2\sqrt{2}}.$$

$$\text{Hence } x = -\frac{1}{\sqrt{2}} \text{Cos}^{-1}(\sqrt{2}y) - \frac{\pi}{2\sqrt{2}}, \quad \text{and}$$

$$\therefore y = \frac{\sqrt{2}}{2} \cos \left(\sqrt{2}x + \frac{\pi}{2} \right).$$

Example 7 Find the general solution to the differential

equation $\frac{dy}{dx} = \cos^2 2y$.

$$\frac{dy}{dx} = \cos^2 2y, \quad \frac{dx}{dy} = \frac{1}{\cos^2 2y} = \sec^2 2y,$$

$$\therefore x = \int \sec^2 2y dy = \frac{1}{2} \tan 2y + c, \quad \therefore \tan 2y = 2(x - c)$$

$$\text{Hence } y = \frac{1}{2} \tan^{-1} 2(x - c).$$

**Numerical solution of $\frac{dy}{dx} = f(x)$ using technology
(analytic solution not required)**

Given $\frac{dy}{dx} = f(x)$ with $y = c$ when $x = a$, the numerical solution of $\frac{dy}{dx} = f(x)$ when $x = b$ is found by evaluating $y = \int_a^b f(x) dx + c$ using graphics calculator.

Example 1 Find the value of y when $x = 0.5$, given

$$\frac{dy}{dx} = \sin(x^2) \text{ where } y = 2 \text{ when } x = 0.$$

$$\frac{dy}{dx} = \sin(x^2),$$

$$\text{when } x = 0.5, \quad y = \int_0^{0.5} \sin(x^2) dx + 2 = 0.0415 + 2 = 2.0415.$$

Example 2 Evaluate Q when $t = 5$, given $\frac{dQ}{dt} = \log_e \left(\frac{10}{t} \right)$ and $Q = 2$ when $t = 1$.

$$\frac{dQ}{dt} = \log_e \left(\frac{10}{t} \right),$$

$$\text{when } t = 1, \quad Q = \int_1^5 \log_e \left(\frac{10}{t} \right) dt + 2 = 5.163 + 2 = 7.163.$$

Example 3 Find the numerical solution to the differential

equation $\frac{dy}{dx} = \sin^{-1} \left(\frac{x}{2} - 1 \right)$ at $x = 3$, given $y = 2$ when $x = 2$.

$$\frac{dy}{dx} = \sin^{-1} \left(\frac{x}{2} - 1 \right),$$

$$\text{when } x = 3, \quad y = \int_2^3 \sin^{-1} \left(\frac{x}{2} - 1 \right) dx + 2 \approx 0.256 + 2 = 2.256.$$

Example 4 Use graphics calculator to solve $\frac{dx}{dt} = \sqrt{1+t^2}$ for x when $t = 1$, given $x = 5$ when $t = 2$.

$$\frac{dx}{dt} = \sqrt{1+t^2},$$

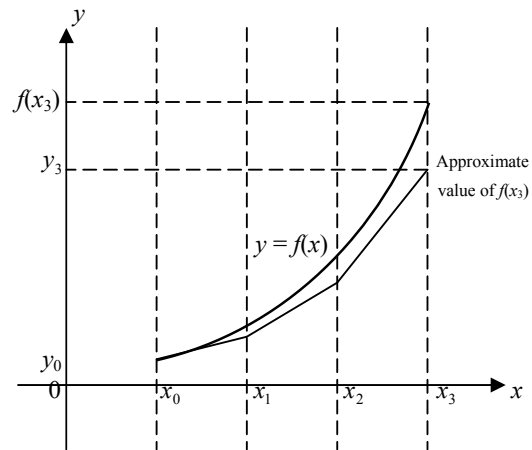
$$\text{when } t = 1, \quad x = \int_2^1 \sqrt{1+t^2} dt + 5 = -\int_1^2 \sqrt{1+t^2} dt + 5 = -1.810 + 5 = 3.190.$$

Euler's method

Euler's method is used to find an approximate numerical solution to a first order differential equation, $\frac{dy}{dx} = F(x, y)$, given $y = y_0$ when $x = x_0$.

The procedure is to find approximate values of y at equally spaced x -coordinates, $x_0, x_1 = x_0 + h, x_2 = x_1 + h, \dots$, where h is the spacing (or step size), using the linear approximation $y_{n+1} \approx y_n + hF(x_n, y_n)$.

The differential equation gives the slope at (x_n, y_n) as $\frac{dy}{dx} = F(x_n, y_n)$.



$$\text{At } x_1, \quad y_1 \approx y_0 + hF(x_0, y_0)$$

$$\text{At } x_2, \quad y_2 \approx y_1 + hF(x_1, y_1)$$

$$\text{At } x_3, \quad y_3 \approx y_2 + hF(x_2, y_2) \text{ etc.}$$

Example 1 Use Euler's method with step size 0.1 to find an approximate solution to $\frac{dy}{dx} = x + y$ when $x = 0.3$, given $y = 1$ when $x = 0$.

$$\frac{dy}{dx} = F(x, y) = x + y.$$

$$\text{At } x = 0, \quad y = 1,$$

$$F(0, 1) = 1.$$

$$\text{At } x = 0.1, \quad y \approx 1 + 0.1 \times 1 = 1.1,$$

$$F(0.1, 1.1) = 1.2.$$

$$\text{At } x = 0.2, \quad y \approx 1.1 + 0.1 \times 1.2 = 1.22,$$

$$F(0.2, 1.22) = 1.42.$$

$$\text{At } x = 0.3, \quad y \approx 1.22 + 0.1 \times 1.42 = 1.362.$$

Example 2 Use Euler's method with step size 20 to find an approximate solution to $\frac{dQ}{dt} = 2 - \frac{9Q}{1000+t}$ at $t = 60$, given $Q = 400$ at $t = 0$. (See example 5 page 8)

$$\frac{dQ}{dt} = F(t, Q) = 2 - \frac{9Q}{1000+t}.$$

$$\text{At } t = 0, \quad Q = 400, \quad F(0, 400) = -1.6.$$

$$\text{At } t = 20, \quad Q = 400 - 20 \times 1.6 = 368, \quad F(20, 368) = -1.247.$$

$$\text{At } t = 40, \quad Q = 368 - 20 \times 1.247 = 343.1, \quad F(40, 343.1) = -0.969.$$

$$\text{At } t = 60, \quad Q = 343.1 - 20 \times 0.969 = 323.7.$$

Setting up differential equations

Example 1 A bacteria culture starts with 1500 bacteria. The population grows to 4500 after 2 hours. Assuming the culture grows at a rate proportional to its population, set up a differential equation for the population growth. Find (a) the constant of proportionality and (b) the population after 5 hours.

(a) Let $N > 0$ be the population at time t hours, and $\frac{dN}{dt}$ be the rate of growth. Assume $\frac{dN}{dt} \propto N$, $\therefore \frac{dN}{dt} = kN$, where k is the constant of proportionality. (Note: k is a positive value for growth; it is a negative value for decline or decay).

$$\frac{dN}{dt} = kN, \therefore \frac{dt}{dN} = \frac{1}{kN}, \therefore t = \int \frac{1}{kN} dN = \frac{1}{k} \int \frac{1}{N} dN,$$

$$\therefore kt = \int \frac{1}{N} dN = \log_e |N| + C = \log_e N + C.$$

When $t = 0$, $N = 1500$, $\therefore C = -\log_e 1500$.

$$\text{Hence } kt = \log_e N - \log_e 1500, \text{ or } kt = \log_e \left(\frac{N}{1500} \right).$$

When $t = 2$, $N = 4500$, $\therefore k = \frac{1}{2} \log_e 3 = \log_e \sqrt{3}$.

$$\therefore (\log_e \sqrt{3})t = \log_e \left(\frac{N}{1500} \right). \text{ Hence } N = 1500e^{t(\log_e \sqrt{3})}.$$

(b) When $t = 5$, $N = 1500e^{5(\log_e \sqrt{3})} = 1500(\sqrt{3})^5 \approx 23400$.

Example 2 In radioactive decay, the rate of decay is proportional to the amount of sample remaining. The half life of radium-226 is 1600 years approximately, i.e. half of the sample remains after about 1600 years. The initial mass of a sample of radium-226 is 200 mg.

- (a) Set up a differential equation for the amount remaining.
 (b) Find a formula for the amount remaining in terms of t years.
 (c) Find the mass after 2000 years.
 (d) When will the mass of radium-226 in the sample be 20 mg?

(a) Let Q mg be the remaining mass of radium-226 after t years, and $\frac{dQ}{dt}$ be the rate of decay. Given $\frac{dQ}{dt} \propto Q$,

$$\therefore \frac{dQ}{dt} = kQ, \text{ where } k \text{ is the proportionality constant.}$$

$$(b) \frac{dQ}{dt} = kQ, \therefore \frac{dt}{dQ} = \frac{1}{kQ}, \therefore t = \int \frac{1}{kQ} dQ = \frac{1}{k} \int \frac{1}{Q} dQ,$$

$$\therefore kt = \int \frac{1}{Q} dQ = \log_e |Q| + C = \log_e Q + C.$$

When $t = 0$, $Q = 200$, $\therefore C = -\log_e 200$.

$$\text{Hence } kt = \log_e Q - \log_e 200, \text{ or } kt = \log_e \left(\frac{Q}{200} \right).$$

When $t = 1600$, $Q = 100$, $\therefore k = \frac{1}{1600} \log_e \frac{1}{2} \approx -0.0004332$.

$$\therefore -0.0004332t = \log_e \left(\frac{Q}{200} \right). \text{ Hence } Q = 200e^{-0.0004332t}.$$

(c) When $t = 2000$, $Q = 200e^{-0.0004332 \times 2000} = 84.09$.

(d) Since $-0.0004332t = \log_e \left(\frac{Q}{200} \right)$,

$$\therefore t = -\frac{1}{0.0004332} \log_e \left(\frac{Q}{200} \right).$$

When $Q = 20$ mg, $t = -\frac{1}{0.0004332} \log_e \left(\frac{20}{200} \right) \approx 5300$ yrs.

Example 3 The rate of cooling of an object is proportional to the difference in temperature between the object and its surroundings. This is known as Newton's law of cooling. Let the room temperature be maintained at 20°C and the object cools from 50 to 30 degrees in 15 minutes.

- (a) Set up a differential equation for the temperature of the object.
 (b) What was the temperature of the object 10 minutes after it was 50°C ?
 (c) When was it at 25°C ?

(a) Let $T > 20^\circ\text{C}$ be the temperature of the object at time t minutes, and $\frac{dT}{dt}$ be the rate of change in temperature. The difference in temperature between the object and its surroundings is $T - 20$.

$$\text{Given } \frac{dT}{dt} \propto (T - 20), \therefore \frac{dT}{dt} = k(T - 20).$$

$$(b) \frac{dT}{dt} = k(T - 20), \therefore \frac{dt}{dT} = \frac{1}{k(T - 20)}.$$

$$\therefore t = \int \frac{1}{k(T - 20)} dT = \frac{1}{k} \int \frac{1}{T - 20} dT,$$

$$\therefore kt = \int \frac{1}{T - 20} dT = \log_e |T - 20| + C = \log_e (T - 20) + C.$$

Let $t = 0$ when $T = 50$, $\therefore C = -\log_e 30$.

$$\text{Hence } kt = \log_e (T - 20) - \log_e 30 = \log_e \left(\frac{T - 20}{30} \right).$$

At $t = 15$, $T = 30$,

$$\therefore 15k = \log_e \left(\frac{1}{3} \right) = -\log_e 3, \therefore k = -\frac{1}{15} \log_e 3 = -0.07324.$$

$$\therefore -0.07324t = \log_e \left(\frac{T - 20}{30} \right) \text{ or } T = 30e^{-0.07324t} + 20.$$

At $t = 10$, $T = 30e^{-0.7324} + 20 = 34.4^\circ\text{C}$.

$$(c) \text{ When } T = 25, -0.07324t = \log_e \left(\frac{25 - 20}{30} \right),$$

$t = 24.5$ minutes.

Example 4 A tank contains 2000 L of brine with a concentration of 0.2 kg of salt per litre. Pure water is run into the tank at a rate of 20 L per minute. The mixture is stirred continuously and allowed to run out at the same rate. (a) Set up a differential equation for the amount of salt in the tank.

- (b) How much salt will remain after half an hour?
 (c) At what time will the concentration be 0.1 kg of salt per litre?

(a) Let Q kg be the amount of salt in the tank at time t minutes, and $\frac{dQ}{dt}$ be the rate of change in the amount of salt in the tank.

$$\frac{dQ}{dt} = \text{rate of inflow of salt} - \text{rate of outflow of salt.}$$

Rate of inflow of salt
 = concentration of solution (kg per L) \times rate of inflow of solution (L per min.)
 = $0 \times 20 = 0$ kg per minute

The volume of solution in the tank remains constant at 2000 L because the volume of inflow equals the volume of outflow. At time t the amount of salt in the tank is Q kg.

$$\therefore \text{concentration} = \frac{Q}{2000} \text{ kg per L}$$

Rate of outflow of salt
 = concentration of solution (kg per L) \times rate of outflow of solution (L per min.)

$$= \frac{Q}{2000} \times 20 = \frac{Q}{100} \text{ kg per minute.}$$

$$\therefore \frac{dQ}{dt} = 0 - \frac{Q}{100}, \text{ i.e. } \frac{dQ}{dt} = -\frac{Q}{100}.$$

$$(b) \therefore \frac{dt}{dQ} = -\frac{100}{Q}, \therefore t = \int \frac{-100}{Q} dQ = -100 \int \frac{1}{Q} dQ,$$

$$\therefore -0.01t = \int \frac{1}{Q} dQ = \log_e |Q| + C = \log_e Q + C.$$

At $t = 0$, $Q = \text{concentration} \times \text{volume} = 0.2 \times 2000 = 400$ kg,

$$\therefore C = -\log_e 400. \text{ Hence } -0.01t = \log_e Q - \log_e 400,$$

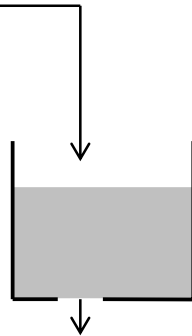
$$-0.01t = \log_e \left(\frac{Q}{400} \right), \therefore Q = 400e^{-0.01t}.$$

At $t = 30$, $Q = 400e^{-0.01 \times 30} = 296$ kg.

(c) When the concentration is 0.1 kg per L,

$$Q = 0.1 \times 2000 = 200 \text{ kg, } \therefore -0.01t = \log_e \left(\frac{200}{400} \right),$$

$t = 69.3$ minutes.



Example 5 Repeat example 4, this time brine with a concentration of 0.1 kg of salt per litre is run into the tank (instead of pure water) at a rate of 20 L per minute, and the mixture runs out at 18 L per minute. Set up a differential equation for the amount of salt in the tank.

Let Q kg be the amount of salt in the tank at time t minutes, and $\frac{dQ}{dt}$ be the rate of change in the amount of salt in the tank.

$$\frac{dQ}{dt} = \text{rate of inflow of salt} - \text{rate of outflow of salt.}$$

Rate of inflow of salt
 = concentration of solution (kg per L) \times rate of inflow of solution (L per min.)
 = $0.1 \times 20 = 2$ kg per minute

Since 20 L per min. of solution runs in and only 18 L per min. of the mixture runs out, the volume of solution in the tank increases by 2 L per min. After t minutes, the volume increases by $2t$ L. \therefore total volume = $2000 + 2t$ L. Hence at

$$\text{time } t, \text{ the concentration} = \frac{Q}{2000 + 2t} \text{ kg per L.}$$

Rate of outflow of salt
 = concentration of solution (kg per L) \times rate of outflow of solution (L per min.)

$$= \frac{Q}{2000 + 2t} \times 18 = \frac{9Q}{1000 + t} \text{ kg per minute}$$

$$\therefore \frac{dQ}{dt} = 2 - \frac{9Q}{1000 + t}.$$

Setting up differential equation involving related rates

Example 1 A spherical snowball is melting at a rate of $r \text{ cm}^3 \text{ min}^{-1}$. (a) Set up a differential equation for its radius r cm. (b) What is its radius after 182π minutes if initially its radius is 10 cm?

(a) Let $V \text{ cm}^3$ be the volume of the spherical snowball of radius r cm at time t minutes, and $\frac{dV}{dt}$ be the rate of change of the volume.

The relationship between V and r is $V = \frac{4}{3}\pi r^3$.

$$\therefore \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = \frac{d}{dr} \left(\frac{4}{3}\pi r^3 \right) \frac{dr}{dt}. \text{ Hence } -r = 4\pi r^2 \frac{dr}{dt}$$

$$\text{and } \therefore \frac{dr}{dt} = -\frac{1}{4\pi r}.$$

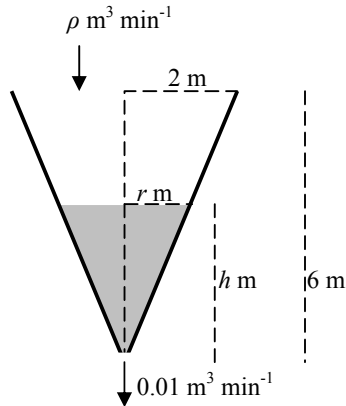
$$(b) \therefore \frac{dt}{dr} = -4\pi r, t = \int -4\pi r dr = -2\pi r^2 + C.$$

Given $r = 10$ when $t = 0$, $\therefore C = 200\pi$.

Hence $t = -2\pi r^2 + 200\pi$.

When $t = 180\pi$, $2\pi r^2 = 20\pi$, $\therefore r = \sqrt{10}$.

Example 2 Water is leaking out of an inverted conical tank at a rate of $0.01 \text{ m}^3 \text{ min}^{-1}$ while water is being pumped into the tank at a constant rate of $\rho \text{ m}^3 \text{ min}^{-1}$. The tank has a height of 6 m, and the radius at the top is 2 m. Initially the water level is 1 m, and it is 4 m after an hour. At what time the level is 5 m?



Let $V \text{ m}^3$ be the volume of water in the tank at time $t \text{ min}$, $h \text{ m}$ the depth and $r \text{ m}$ the radius of the water surface.

Rate of change in volume = inflow rate – outflow rate,

$$\text{i.e. } \frac{dV}{dt} = \rho - 0.01$$

The relationship between h and r is given by $\frac{r}{h} = \frac{2}{6}$,

$\therefore r = \frac{h}{3}$. Hence the relationship between V and h is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}h^3.$$

$$\therefore \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}. \quad \therefore \frac{dh}{dt} = \frac{9(\rho - 0.01)}{\pi h^2}.$$

$$\frac{dt}{dh} = \frac{\pi h^2}{9(\rho - 0.01)}, \quad t = \int \frac{\pi h^2}{9(\rho - 0.01)} dh = \frac{\pi h^3}{27(\rho - 0.01)} + C.$$

$$\text{At } t = 0, h = 1, \therefore C = -\frac{\pi}{27(\rho - 0.01)}, \quad \therefore t = \frac{\pi(h^3 - 1)}{27(\rho - 0.01)}.$$

$$\text{At } t = 60, h = 4, \quad 60 = \frac{63\pi}{27(\rho - 0.01)}, \quad \therefore \rho = 0.1322.$$

$$\text{When } h = 5, \quad t = \frac{\pi(5^3 - 1)}{27(0.1322 - 0.01)} = 118 \text{ minutes.}$$