Kinematics and Mechanics

Rectilinear motion

Kinematics is the study of motion. Rectilinear motion means motion in a straight line (or one-dimensional motion). In rectilinear motion there are only two directions. If one direction is labelled as positive, the opposite direction is negative.

Position

Use the number line to describe **position** *x*. Position is a vector (starting from the origin O) and a positive (usually pointing to the right or up) or negative (to the left or down) sign is used to indicate its direction.

Displacement

When an object moves, the change in position is called its displacement. Displacement is defined as $s = \Delta x = x_f - x_i$.

Distance travelled

Distance travelled, *l*, is the actual length of the path followed by the object.

Average velocity

Average velocity is defined as $\frac{displacement}{timetaken}$, i.e. $v_{av} = \frac{\Delta x}{\Delta t}$.
It is a vector quantity.

Average speed

Average speed is defined as $\frac{dis\tan cetravelled}{timetaken}$, i.e. $v_{av} = \frac{l}{\Delta t}$.
It is a scalar quantity.

Instantaneous velocity (i.e. velocity)

If position is given as a function of time, $x(t)$, then velocity is defined as $v = \frac{dx}{dt}$.
The direction of velocity is the direction of motion.

Speed is given by the magnitude of velocity, i.e. $v = \left|\frac{dx}{dt}\right|$.

Note that average speed is not equal to the magnitude of average velocity in general.

Acceleration

Acceleration is defined as $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$. It is a vector quantity, and its direction shows the direction that the resultant (net) force acts.

Note that the direction of motion does not correspond to the direction of acceleration in general.

Example 1 A particle moves in a straight line such that its position at time *t* is $x = 2\left(1 - e^{-\frac{t}{2}}\right)$, $t \geq 0$.

(a) Find the particle's velocity at $t = 1$.
(b) Find the particle's velocity at $x = 1$.

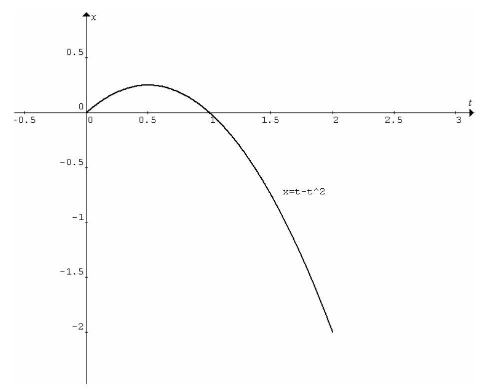
(a) $x = 2\left(1 - e^{-\frac{t}{2}}\right)$, $v = \frac{dx}{dt} = e^{-\frac{t}{2}}$. At $t = 1$, $v = e^{-\frac{1}{2}}$.

(b) At $x = 1$, $1 = 2\left(1 - e^{-\frac{t}{2}}\right)$, $\therefore e^{-\frac{t}{2}} = \frac{1}{2}$, $\therefore v = \frac{1}{2}$.

Example 2 The position of an object is given by $x = t - t^2$, $t \geq 0$.

(a) Sketch the graph of *x* vs *t* and describe the motion in terms of displacement.
(b) Find the total distance travelled from *t* = 0 to *t* = 2.
(c) Calculate the average velocity and the average speed from $t = 0$ to $t = 2$.
(d) Calculate the velocity and the speed at *t* = 0.2.
(e) Calculate the acceleration.

(a)



The object starts from the origin and moves forwards. At $t = 0.5$, its position is 0.25 and it starts to move backwards. It keeps on moving backwards, passing through the starting point (the origin) at $t = 1$.

(b) At $t = 2$, $x = -2$, $\therefore$total distance $= 0.25 + 2.25 = 2.5$ where 0.25 is the forward distance and 2.25 is the backward distance.

(c) Average velocity $v_{av} = \frac{\Delta x}{\Delta t} = \frac{-2 - 0}{2} = -1$.

Average speed $v_{av} = \frac{l}{\Delta t} = \frac{2.5}{2} = 1.25$

(d) Velocity $v = \frac{dx}{dt} = 1 - 2t = 1 - 4 = -3$. Speed $v = \left|\frac{dx}{dt}\right| = 3$.

(e) Acceleration $a = \frac{dv}{dt} = -2$.

Example 3 The position x of a particle at time t is given by $x = 3\sin 2t$, $t \ge 0$.

(a) Find the position and the speed of the particle at $t = 0$.
(b) Find the maximum displacement of the particle.
(c) Find the maximum speed of the particle.
(d) Show that the acceleration is opposite in direction and proportional to the displacement.
(e) After what time will the particle repeat its motion?
(f) When is the first time that the particle is at the position $x = 1.5$ moving towards the starting point?

(a) Position $x = 3\sin 2t$, speed $v = \left|\frac{dx}{dt}\right| = |6\cos 2t|$.

At $t = 0$, position $x = 0$, speed $v = 6$.

(b) Displacement $s = \Delta x = x_f - x_i = 3\sin 2t - 0 = 3\sin 2t$.

Since $-1 \le \sin 2t \le 1$, max $s = 3$.

(c) Speed $v = |6\cos 2t|$. Since $-1 \le \cos 2t \le 1$, max $v = 6$.

(d) Velocity $v = \frac{dx}{dt} = 6\cos 2t$,

acceleration $a = \frac{dv}{dt} = -12\sin 2t$.

At time t, $s = 3\sin 2t$, $\therefore a = -4s$.

Hence a is opposite (negative sign) to s and it is proportional to s.

(e) The type of motion that a is opposite to s and proportional to s is called simple harmonic motion. The motion is repeated after a fixed time called the period $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$.

(f) $x = 3\sin 2t$, $3\sin 2t = 1.5$, $\sin 2t = 0.5$,

$2t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, ...$ or $t = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, ...$

At $t = \frac{\pi}{12}$, velocity $v = 6\cos 2\left(\frac{\pi}{12}\right) = +3\sqrt{3}$, the particle moves away from the starting point.

At $t = \frac{5\pi}{12}$, velocity $v = 6\cos 2\left(\frac{5\pi}{12}\right) = -3\sqrt{3}$, the particle moves towards the starting point.

Hence the first time the particle is at position $x = 1.5$ moving towards the starting point is $t = \frac{5\pi}{12}$.

Example 4 Given the position of a particle $x = 2te^{-t}$ for $0 \le t \le 1.5$,

(a) find the position and the velocity at $t = 0$, and
(b) find the maximum and minimum velocity and the time that each occurs.
(c) Find the minimum speed and the time that it occurs.
(d) Sketch the position-time graph for $0 \le t \le 1.5$.

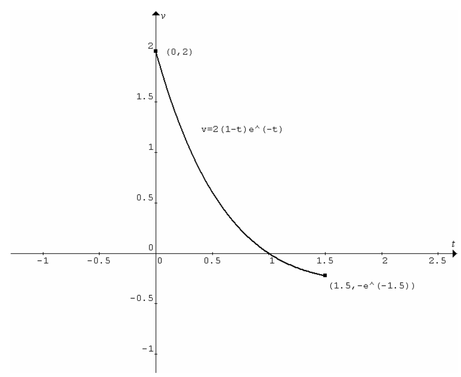
(a) $x = 2te^{-t}$, $v = \frac{dx}{dt} = -2te^{-t} + 2e^{-t} = 2e^{-t}(1 - t)$.

At $t = 0$, $x = 0$, $v = 2$.

(b) Local max/min velocity occurs when $a = \frac{dv}{dt} = 0$.

$\therefore a = -2e^{-t}(1 - t) - 2e^{-t} = 2e^{-t}(t - 2) = 0$, i.e. $t = 2$, which is outside the domain $0 \le t \le 1.5$ that $x = 2te^{-t}$ is defined.

Within $0 \le t \le 1.5$, velocity $v = 2e^{-t}(1 - t)$ is a decreasing function (see the following graph of velocity $v(t)$). Hence max. velocity $v_{\max} = 2$ occurs at $t = 0$, and min. velocity $v_{\min} = -e^{-1.5}$ occurs at $t = 1.5$.

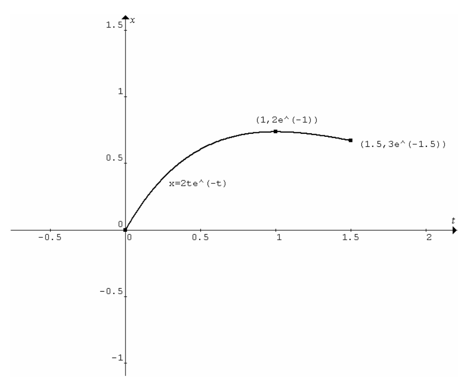


Note: Students are advised to analyse the velocity-time graph of motion before using calculus to find local maxima and minima.

(c) Speed is the magnitude of velocity. From the graph above, the particle comes to a stop momentarily at $t = 1$. Hence the minimum speed is 0.

(d) At $t = 1.5$, $x = 2(1.5)e^{-1.5} = 3e^{-1.5}$. Position x is maximum when velocity $v = 2e^{-t}(1 - t) = 0$, i.e. $t = 1$.

$\therefore x_{\max} = 2(1)e^{-1} = 2e^{-1}$.



Anti-differentiation of acceleration and velocity

If acceleration is a known function of time, $a(t)$, then velocity $v(t)=\int a\,dt$ and position $x(t)=\int v\,dt$.

Change in velocity in the time interval from t_1 to t_2 is given by

$$\Delta v=\int_{t_1}^{t_2} a\,dt.$$

Displacement in the time interval from t_1 to t_2 is given by

$$s=\int_{t_1}^{t_2} v\,dt.$$

Example 1 An object is projected vertically upwards from a height of 8 m above the ground with a speed of 12 ms^{-1}. Take the acceleration due to gravity as 10 ms^{-2} downward and $t=0$ at the time of projection.

(a) Determine a formula for finding the velocity at time $t\geq 0$.
(b) Determine a formula for finding the position at time *t*.
(c) Determine a formula for finding the displacement at time *t*.
(d) Find the time when the object comes to a stop momentarily.
(e) Find the maximum height of object above the ground.
(f) Find the time when the object hits the ground.

Choose upward direction as positive, ground level as the origin.

(a) Velocity $v(t)=\int a\,dt=\int -10\,dt=-10t+C_1$.

At $t=0$, velocity $v=+12$, $\therefore C_1=12$.

$\therefore v(t)=-10t+12$.

(b) Position $x(t)=\int v\,dt=\int -10t+12\,dt=-5t^2+12t+C_2$.

At $t=0$, position $x=+8$, $\therefore C_2=8$.

$\therefore x=-5t^2+12t+8$.

(c) Displacement in the time interval from 0 to *t*:

$$s=\int_{t_1}^{t_2} v\,dt=\int_0^t(-10t+12)dt=\left[-5t^2+12t\right]_0^t=-5t^2+12t.$$

(d) When the object comes to a stop, $v(t)=-10t+12=0$, $t=1.2$ s.

(e) At $t=1.2$, the object is at its highest point

$x=-5t^2+12t+8=-5(1.2)^2+12(1.2)+8=15.2$

Maximum height = 15.2 m

(f) When the object hits the ground, $x=-5t^2+12t+8=0$, $t\approx 2.9$ s.

Example 2 A particle's variable acceleration is given by $a=1-e^{-t}$, $t\geq 0$. The particle is initially at rest.

(a) Find the average acceleration of the particle in the time interval from $t=1$ to $t=2$.
(b) Find the average velocity of the particle in the time interval from $t=1$ to $t=2$.

(a) $\Delta v=\int_{t_1}^{t_2} a\,dt=\int_1^2\left(1-e^{-t}\right)dt=\left[t+e^{-t}\right]_1^2$

$=2+e^{-2}-1-e^{-1}=1+e^{-2}-e^{-1}$.

Average acceleration $a_{av}=\dfrac{\Delta v}{\Delta t}=\dfrac{1+e^{-2}-e^{-1}}{2-1}=1+e^{-2}-e^{-1}$.

(b) $v(t)=\int a\,dt=\int\left(1-e^{-t}\right)dt=t+e^{-t}+C$. At $t=0$, $v=0$, $\therefore C=-1$, $\therefore v=t+e^{-t}-1$.

$$s=\int_{t_1}^{t_2} v\,dt=\int_1^2\left(t+e^{-t}-1\right)dt=\left[\frac{t^2}{2}-e^{-t}-t\right]_1^2$$

$$=\left(2-e^{-2}-2\right)-\left(\frac{1}{2}-e^{-1}-1\right)=\frac{1}{2}-e^{-2}+e^{-1}.$$

Average velocity $v_{av}=\dfrac{s}{\Delta t}=\dfrac{\frac{1}{2}-e^{-2}+e^{-1}}{2-1}=\dfrac{1}{2}-e^{-2}+e^{-1}$.

Motion under constant acceleration

Consider an object moving with constant acceleration *a*. Its initial velocity is *u* and its initial position is x_i. Let *v* be its velocity at time *t*, and *s* be its displacement from the initial position at time *t*.

$v=\int a\,dt=at+C_1$.

At $t=0$, $v=u$, $\therefore C_1=u$, $\therefore v=u+at$ ……… (1)

$x=\int v\,dt=\int(u+at)dt=ut+\frac{1}{2}at^2+C_2$.

At $t=0$, $x=x_i$, $\therefore C_2=x_i$, $\therefore x=ut+\frac{1}{2}at^2+x_i$.

Since $s=x-x_i$, $\therefore s=ut+\frac{1}{2}at^2$ ……… (2)

From (1), $u=v-at$, substitute in (2) to obtain

$s=vt-\frac{1}{2}at^2$ ……… (3)

(2) + (3), $s=\frac{1}{2}(u+v)t$ ……… (4)

From (1), $t=\dfrac{v-u}{a}$, substitute in (4), transpose to obtain

$v^2=u^2+2as$ ……… (5)

These five equations are used instead of calculus to analyse rectilinear motion with constant acceleration. Each equation involves 4 out of the 5 quantities, *s*, *t*, *u*, *v* and *a*.
s is the displacement at time *t* from the initial position (not necessarily the origin). *v* is the velocity at time *t* (sometimes called the final velocity).
Since *s*, *u*, *v* and *a* are vector quantities, directions (+, −) are required to specify them completely in the equations.

Example 1 A particle changes its velocity from 5 ms^{-1} to 10 ms^{-1} after travelling 100 m with constant acceleration.
(a) Find the acceleration of the particle.
(b) Find the time taken.

Forward direction is taken as positive.
$u=+5,\ v=+10,\ s=+100,\ a=?$ Use $v^2=u^2+2as$.
$a=\dfrac{v^2-u^2}{2s}=\dfrac{100-25}{200}=+0.375$ ms^{-2}.

$u=+5,\ v=+10,\ s=+100,\ t=?$ Use $s=\dfrac{1}{2}(u+v)t$.
$t=\dfrac{2s}{u+v}=\dfrac{200}{15}=13.3$ s.

Example 2 A particle moves with constant acceleration. Its velocity reduces from 10 ms^{-1} to 5 ms^{-1} in 2 s.
(a) Find the distance it travels before coming to rest momentarily.
(b) How long does it take to return to its initial position.
(c) Find the time(s) when it is 2 m from the initial position.

Forward direction is taken as positive.
(a) $u=+10,\ v=+5,\ t=2,\ a=?$ Use $v=u+at$.
$a=\dfrac{v-u}{t}=\dfrac{-5}{2}=-2.5$ s.
$u=+10,\ v=0,\ a=-2.5,\ s=?$ Use $v^2=u^2+2as$.
$s=\dfrac{v^2-u^2}{2a}=\dfrac{0-100}{-5}=+20$ m. Distance = 20 m.
(b) $u=+10,\ a=-2.5,\ s=0,\ t=?$ Use $s=ut+\dfrac{1}{2}at^2$.
$0=10t-1.25t^2,\ (1.25t-10)t=0,\ \therefore t=8$ s.
Time required = 8 s.
(c) $u=+10,\ a=-2.5,\ s=\pm 2,\ t=?$ Use $s=ut+\dfrac{1}{2}at^2$.
$+2=10t-1.25t^2,\ 1.25t^2-10t+2=0,\ t\approx 7.8$ s.
$-2=10t-1.25t^2,\ 1.25t^2-10t-2=0,\ t\approx 8.2$ s.

Example 3 An object is projected vertically upwards from a bridge. It reaches the water below after 2 seconds at a speed double the speed of projection.
(a) Find the speed of projection.
(b) The height of the bridge above the water.

Let V ms^{-1} be the speed of projection, and h m the height of the bridge above the water.
Take upward direction as positive.
(a) $u=+V,\ v=-2V,\ a=-10,\ t=2.$ Use $v=u+at$.
$-2V=+V-20,\ V=\dfrac{20}{3}$ ms^{-1}.
(b) $u=+V,\ v=-2V,\ a=-10,\ s=-h.$ Use $v^2=u^2+2as$.
$(-2V)^2=(+V)^2+2(-10)(-h),\ 4V^2=V^2+20h,\ 20h=3V^2,$
$h=\dfrac{3}{20}\left(\dfrac{20}{3}\right)^2=\dfrac{20}{3}$ metres.

Two other derivative forms for acceleration

Acceleration a can also be expressed in terms of position x and velocity v, $a=v\dfrac{dv}{dx}$ or $a=\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)=\dfrac{1}{2}\dfrac{d}{dx}v^2$.

These different derivative forms for acceleration can be obtained by means of the chain rule.

$$a=\frac{dv}{dt}=\frac{dv}{dx}\times\frac{dx}{dt}=v\frac{dv}{dx},$$
$$a=v\frac{dv}{dx}=\frac{d\left(\frac{1}{2}v^2\right)}{dv}\times\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right).$$

Example 1 The motion of a particle is described by $v^2=4x-3x^2$ where v ms^{-1} is the velocity when the particle is at position x m.
(a) Find the acceleration at position x.
(b) Find its maximum speed.

(a) $a=\dfrac{1}{2}\dfrac{d}{dx}v^2=\dfrac{1}{2}(4-6x)=2-3x$.

(b) Speed is maximum when $a=0,\ 2-3x=0,\ x=\dfrac{2}{3}$.

$\therefore v_{max}=\sqrt{4\left(\dfrac{2}{3}\right)-3\left(\dfrac{2}{3}\right)^2}=\dfrac{2\sqrt{3}}{3}$ ms^{-1}.

Example 2 The motion of a particle is described by $v=\dfrac{x}{2}-2$.
Find the acceleration in terms of its position x.

$a=v\dfrac{dv}{dx}=\left(\dfrac{x}{2}-2\right)\dfrac{d}{dx}\left(\dfrac{x}{2}-2\right)=\left(\dfrac{x}{2}-2\right)\dfrac{1}{2}=\dfrac{x}{4}-1$.

Example 3 The acceleration of a particle is related to its velocity according to $a=v^2-1$. If $v=-2$ when $x=0$, find v in terms of its position x.

Both derivative forms can be used in this example.
(1) $a=v^2-1,\ \dfrac{1}{2}\dfrac{d(v^2)}{dx}=v^2-1,\ \dfrac{d(v^2)}{dx}=2(v^2-1),$
$\dfrac{dx}{d(v^2)}=\dfrac{1}{2}\dfrac{1}{v^2-1},\ x=\dfrac{1}{2}\displaystyle\int\dfrac{1}{(v^2)-1}d(v^2),\ \therefore 2x=\log_e|v^2-1|+C.$
(2) $a=v^2-1,\ v\dfrac{dv}{dx}=v^2-1,\ \dfrac{dv}{dx}=\dfrac{v^2-1}{v},\ \dfrac{dx}{dv}=\dfrac{v}{v^2-1}.$
Hence $x=\displaystyle\int\dfrac{v}{v^2-1}dv.$ Substitute u for v^2-1, $x=\dfrac{1}{2}\displaystyle\int\dfrac{1}{u}du,$
$\therefore 2x=\log_e|v^2-1|+C,$ the same result as in (1).
When $x=0,\ v=-2,\ \therefore C=-\log_e 3.$
$\therefore 2x=\log_e|v^2-1|-\log_e 3=\log_e\dfrac{|v^2-1|}{3}.$ Hence
$|v^2-1|=3e^{2x},\ \therefore v^2-1=\pm 3e^{2x}.$ Only $v^2-1=3e^{2x}$ satisfies the boundary condition that when $x=0,\ v=-2$.
$\therefore v^2=3e^{2x}+1,\ v=\pm\sqrt{3e^{2x}+1}.$ Only $v=-\sqrt{3e^{2x}+1}$ satisfies the boundary condition that when $x=0,\ v=-2$.

Example 4 The acceleration of a particle is given by $1-x$, where x is its position at time t. At $t = 0$, $x = 1$ and $v = 1$.

(a) Find its velocity v in terms of x.
(b) Find its position x as a function of t.
(c) Find its velocity and acceleration in terms of t.

(a) $a = 1-x,\ \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1-x,$

$\frac{1}{2}v^2 = \int (1-x)dx = x - \frac{1}{2}x^2 + C.$

At $x = 1$, $v = 1$, $\therefore C = 0$ and $\frac{1}{2}v^2 = x - \frac{1}{2}x^2.$

Hence $v = \pm\sqrt{2x-x^2}$. Only $v = \sqrt{2x-x^2}$ satisfies the boundary condition that at $x = 1$, $v = 1$.

(b) $v = \sqrt{2x-x^2}$, $\frac{dx}{dt} = \sqrt{2x-x^2}$, $\frac{dt}{dx} = \frac{1}{\sqrt{2x-x^2}}$, completing the square for $2x-x^2$, $\frac{dt}{dx} = \frac{1}{\sqrt{1-(x-1)^2}}$,

$t = \int \frac{1}{\sqrt{1-(x-1)^2}}dx,\ \therefore t = Sin^{-1}(x-1) + D.$ At $t = 0$, $x = 1$,

$\therefore D = 0$ and $t = Sin^{-1}(x-1)$. Hence $x = \sin t + 1$.

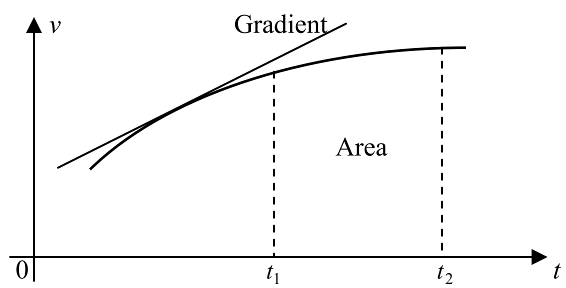
(c) $x = \sin t + 1,\ v = \frac{dx}{dt} = \cos t,\ a = \frac{dv}{dt} = -\sin t.$

Check: $a = -\sin t$ and $x = \sin t + 1$, $\therefore a = 1-x$ as given.

Velocity-time graphs

Gradient of a velocity-time graph gives acceleration, i.e. $\frac{dv}{dt} = a$.

Area under a velocity-time graph gives displacement, i.e. $\int_{t_1}^{t_2} v dt = s$.



Example 1 A tram runs on a straight track between two stations 1.6 km apart. It accelerates from rest from one station at 0.20 ms^{-2} until it reaches a speed of 10 ms^{-1}. It maintains this speed for a time and then slows to rest at the other station at 0.30 ms^{-2}. Find the time of travel between the two stations.

The time for the tram to accelerate from rest to 10 ms^{-1}:
$u = 0,\ v = +10,\ a = +0.20,\ t = ?$

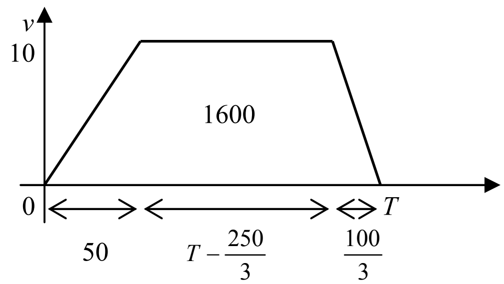
Use $v = u + at,\ \therefore t = \frac{v-u}{a} = 50$ s.

The time for the tram to decelerate from 10 ms^{-1} to a stop:
$u = +10,\ v = 0,\ a = -0.30,\ t = ?$

Use $v = u + at,\ \therefore t = \frac{v-u}{a} = \frac{100}{3}$ s.

Let T s be the time of travel between the two stations. Then $T - 50 - \frac{100}{3} = T - \frac{250}{3}$ is the duration that the tram is at constant speed 10 ms^{-1}.

Construct the velocity-time graph for the journey.



Area under v-t graph = displacement.

$\frac{1}{2}\left(T - \frac{250}{3} + T\right)\times 10 = 1600,\ 2T - \frac{250}{3} = 320,\ 2T = \frac{1210}{3},$

$\therefore T = \frac{605}{3}$ s.

Example 2 A tram moves with an acceleration of $\frac{2}{625}t(t^2-100)$ ms^{-2} at time $t \ge 0$. At $t = 0$, its speed is 8 ms^{-1}.

(a) Find the time required for the tram to stop.
(b) Find the distance travelled for it to stop.

(a) At time t the tram moves with a velocity of

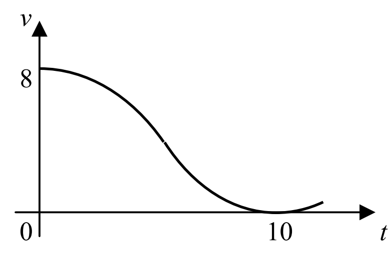
$v = \int_0^t \frac{2}{625}t(t^2-100)dt + 8 = \left[\frac{2}{625}\left(\frac{t^4}{4} - 50t^2\right)\right]_0^t + 8$

$= \frac{t^4}{1250} - \frac{4t^2}{25} + 8 = \frac{1}{1250}(t^4 - 200t^2 + 10000) = \frac{1}{1250}(t^2-100)^2$

$= \frac{1}{1250}(t-10)^2(t+10)^2.\ \therefore$ at $t = 10,\ v = 0.$

Time required is 10 s.

Sketch the v-t graph for $t \ge 0$.



(b) Area under v-t graph = displacement $= \int_0^{10} v dt$

$= \int_0^{10} \frac{1}{1250}(t^4 - 200t^2 + 10000)dt$

$= \left[\frac{1}{1250}\left(\frac{t^5}{5} - \frac{200t^3}{3} + 10000t\right)\right]_0^{10} = \frac{128}{3}.$

Distance travelled $= \frac{128}{3}$ m.

Mechanics

In mechanics we study the cause for change in motion. Change in motion of a particle is the result of some forces acting on it. The relationship between force and motion is described in Newton's second law. If the forces on a particle have zero vector sum, its motion remains constant, i.e. it remains at rest if it is at rest; it continues to move in the same direction at the same speed if it is moving. This is known as Newton's first law (or the law of inertia). A particle is in **static equilibrium** when it remains at rest, the vector sum of forces equals zero. It is in **limiting equilibrium** when it is at rest but on the verge of moving.

Newton's second law

Newton's second law, $a=\frac{R}{m}$ where a ms^{-2} is the acceleration of a particle of mass m kg, and R newtons (N) is the resultant force on it.
Both a and R are vectors pointing in the same direction.

Resultant force R is the vector addition of all forces acting on the particle, $R=F_1+F_2+F_3+......$

R can be resolved into two perpendicular components, $R_x=F_{1x}+F_{2x}+F_{3x}+......$ and $R_y=F_{1y}+F_{2y}+F_{3y}+......$

Newton's second law $R=ma$ can be expressed in a more general way, $R=\frac{dp}{dt}$ where p is a vector quantity called momentum (kgms^{-1}) of a particle, it is defined as $p=mv$, the product of mass and velocity. $R=ma$ is the special case when m is constant.

If R is constant, then a is constant and $R=m\frac{\Delta v}{\Delta t}=\frac{\Delta p}{\Delta t}$. If R is not constant, then average resultant force $R_{av}=\frac{\Delta p}{\Delta t}$.

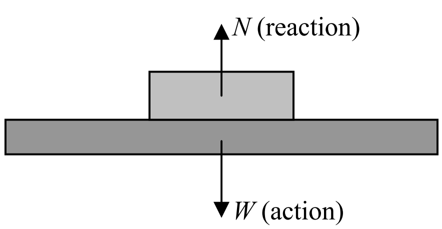
Weight

The weight W (N) of an object is the force of gravity on it. Weight is a vector and its direction is always downward. Its magnitude is given by $W=mg$ and g has a value of 9.8 Nkg^{-1}.

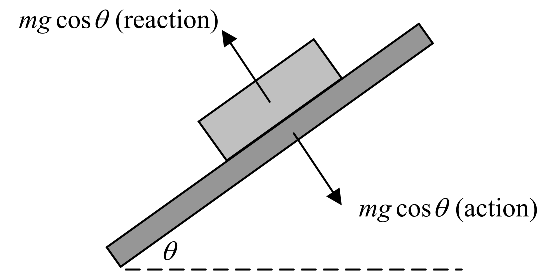
Newton's third law

Newton's third law states that for each action there is a reaction, e.g. when an object rests or slides along a smooth horizontal surface, it presses against the surface with a force (action) equals to its weight mg newtons, the surface therefore exerts a force (reaction) of the same magnitude on the object but in the opposite direction.

In this case the reaction force is perpendicular to the surface and it is called normal reaction N.



If the object slides along a *smooth* plane inclined at an angle to the horizontal, the object presses against the plane with a force (action) equals to $mg\cos\theta$ and therefore the plane exerts a reaction force of the same magnitude on the object. Both action and reaction are normal to the plane if there is no friction force.

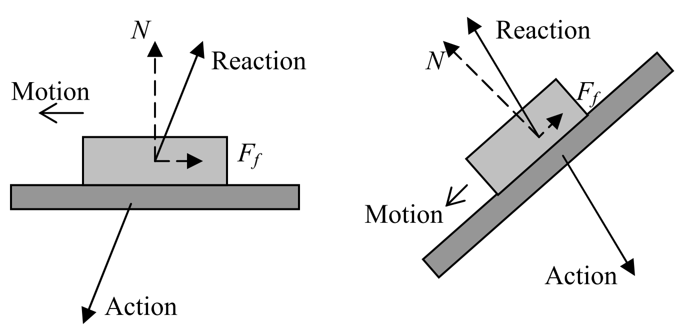


Friction

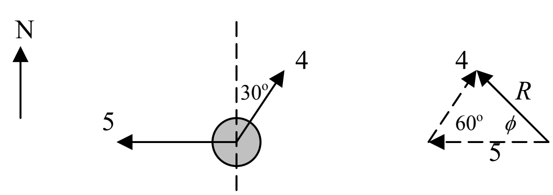
Friction exists between two sliding surfaces in contact. It provides a force (called force of friction) that always acts in the opposite direction to motion (or tendency of motion). If an object is at rest, force of friction can range from 0 to a maximum value given by $F_f=\mu N$, where N is the magnitude of the normal reaction of the surface on the object and μ is a numerical value called the **coefficient of friction**. The value of μ depends on the two surfaces in contact.

Friction reaches its maximum value when the object is on the verge of sliding, or when it is sliding along the surface.

If there is a force of friction, the reaction force of the surface on the object is not normal to the surface. It consists of two components: the normal reaction N and the force of friction F_f.



Example 1 A particle is subject to two forces, one of 5 newtons acting due west, the other of 4 newtons acting at a bearing of $N30^0E$. Calculate the resultant force on the particle.



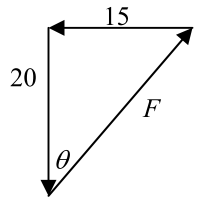
Use the cosine rule: $|R|^2=4^2+5^2-2(4)(5)\cos 60^o$,

$|R|=\sqrt{21}\approx 4.6$. Use the sine rule: $\frac{4}{\sin\phi}=\frac{\sqrt{21}}{\sin 60^o}$, $\phi\approx 49^o$.

The resultant force is approx. 4.6 newtons N41^{o}W.

Example 2 Two forces act on a particle. One is 20 N vertically downward and the other is 15 N horizontally to the left. Find the third force that keeps the particle in equilibrium.

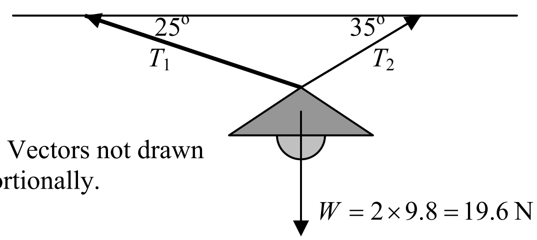
The particle is in equilibrium, $\therefore$ the resultant force $R = 0$. The three forces form a closed triangle. Let F be the third force.



$|F| = \sqrt{15^2 + 20^2} = 25,\ \theta = \tan^{-1}\left(\frac{15}{20}\right) \approx 37^o$.

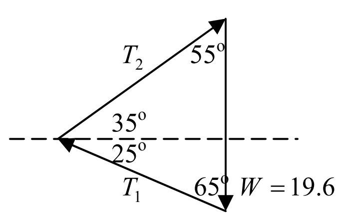
The third force to balance the other two is 25 newtons at approx. 37^o with the vertical.

Example 3 Find the tension in the electric cord (T_1) and in the string (T_2) that holds the 2-kg lamp as shown in the following diagram.



Note: Vectors not drawn proportionally.

The lamp is in equilibrium, $\therefore$ resultant force = 0, $\therefore$ the three forces T_1, T_2 and W form a closed triangle.



Find the angles and use the sine rule:

$\frac{|T_1|}{\sin 55^o} = \frac{|T_2|}{\sin 65^o} = \frac{19.6}{\sin 60^o}$. $\therefore |T_1| \approx 18.5$ N and $|T_2| \approx 20.5$ N.

Alternative method: Resolve vectors into vertical and horizontal components.

From the first diagram, $-|T_1|\cos 25^o + |T_2|\cos 35^o = 0$,

$|T_1|\sin 25^o + |T_2|\sin 35^o - 19.6 = 0$.

Solve the two equations simultaneously to find $|T_1|$ and $|T_2|$.

Example 4 A particle of mass 2 kg has an initial velocity of $-12\,\mathbf{i}$ ms^{-1}. A constant force acts on the particle so that after 2 s its velocity is $16\,\mathbf{j}$ ms^{-1}.

(a) Calculate the change in momentum of the particle.
(b) Calculate the constant force causing the change in velocity.

(a) $\Delta p = mv - mu = 2\,(16\,\mathbf{j}) - 2\,(-12\,\mathbf{i}) = 24\,\mathbf{i} + 32\,\mathbf{j}$.

(b) $R = \frac{\Delta p}{\Delta t} = (24\,\mathbf{i} + 32\,\mathbf{j})/2 = 12\,\mathbf{i} + 16\,\mathbf{j}$.

Example 5 The momentum of a spacecraft at time t is given by $p(t) = 2\,\mathbf{i} + t^2\,\mathbf{j} - t\,\mathbf{k}$.

(a) Calculate the change in momentum from $t = 0$ to $t = 5$.
(b) Calculate the average resultant force from $t = 0$ to $t = 5$.
(c) Calculate the resultant force at $t = 5$.

(a) $p(0) = 2\,\mathbf{i},\ p(5) = 2\,\mathbf{i} + 25\,\mathbf{j} - 5\,\mathbf{k},\ \Delta p = p(5) - p(0) = 25\,\mathbf{j} - 5\,\mathbf{k}$.

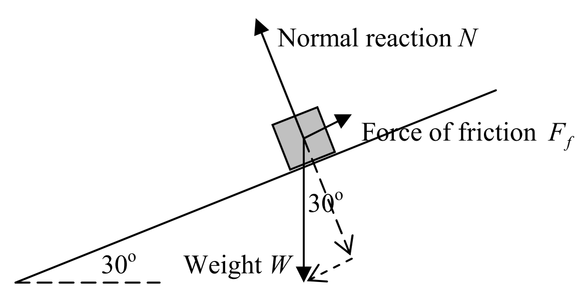
(b) $R_{av} = \frac{\Delta p}{\Delta t} = (25\,\mathbf{j} - 5\,\mathbf{k})/5 = 5\mathbf{j} - \mathbf{k}$.

(c) $R = \frac{dp}{dt} = 2t\,\mathbf{j} - \mathbf{k}$. At $t = 5,\ R = \mathbf{10\ j} - \mathbf{k}$.

Example 6 A girl slides down a 6-m long plane inclined at an angle of 30^o to the horizontal. The girl has mass of 30 kg. The coefficient of friction between the girl and the slide is 0.2.

(a) Analyse the forces acting on the girl as she slides down.
(b) Determine her acceleration in terms of g.
(c) She starts from rest at the top of the slide. Find the time required to reach the end of the slide.
(d) Determine her momentum at the end of the slide.

(a)



$|W| = mg = 30g$,

$|N| - |W|\cos 30^o = 0,\ \therefore |N| = |W|\cos 30^o = 30g\cos 30^o = 15\sqrt{3}g$.

$|F_f| = \mu|N| = 0.2 \times 15\sqrt{3}g = 3\sqrt{3}g$.

(b) Use Newton's second law for component along the slide:

$$a = \frac{30g\sin 30^o - 3\sqrt{3}g}{30} = \left(\frac{5-\sqrt{3}}{10}\right)g .$$

(c) $u = 0,\ s = +6,\ a = +\left(\frac{5-\sqrt{3}}{10}\right)g,\ t = ?$ Use $s = ut + \frac{1}{2}at^2$,

$$6 = \frac{1}{2}\left(\frac{5-\sqrt{3}}{10}\right)gt^2,\ t = 1.9 \text{ s}$$

(d) $u = 0,\ s = +6,\ a = +\left(\frac{5-\sqrt{3}}{10}\right)g,\ v = ?$ Use $v^2 = u^2 + 2as$,

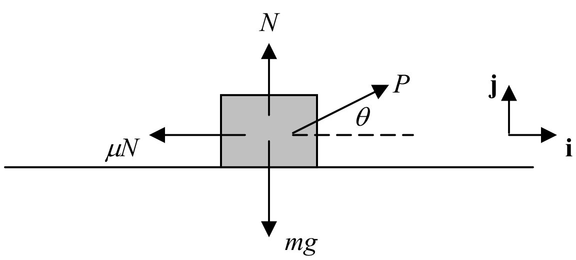
$v = \sqrt{2\left(\frac{5-\sqrt{3}}{10}\right)g(6)} = 6.2\ \text{ms}^{-1}$.

$\therefore p = mv = 30 \times 6.2 = 186\ \text{kgms}^{-1}$.

Example 7 A crate of mass m kg rests on a rough, level ground. It is pulled with a force of magnitude P newtons tilted upwards at an angle of θ to the horizontal. There is a normal reaction of magnitude N newtons and the coefficient of friction between the box and the ground is μ. The crate is on the verge of sliding along the ground.

(a) Express N in terms of g, P, m and θ.
(b) Express μ in terms of g, P, m and θ.
(c) Determine the magnitude of the reaction of the ground on the crate in terms of g, P, m and θ.
(d) Show that the reaction force inclines at angle $\phi = \tan^{-1}\mu$ with the vertical. Hence express ϕ in terms of g, P, m and θ.
(e) Find the relationship among g, P, m and θ for the crate to remain on the ground.

Since the crate is on the verge of sliding, force of friction is at maximum value μN.



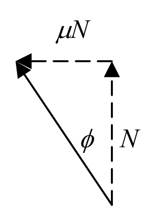
i-component: $P\cos\theta - \mu N = 0$ ………(1)
j-component: $P\sin\theta + N - mg = 0$ ……(2)

(a) From (2), $N = mg - P\sin\theta$ ……(3)

(b) Substitute (3) in (1), $P\cos\theta - \mu(mg - P\sin\theta) = 0$,

$\therefore \mu = \frac{P\cos\theta}{mg - P\sin\theta}$.

(c) Magnitude of reaction



$= \sqrt{(\mu N)^2 + (mg - P\sin\theta)^2}$

$= \sqrt{(P\cos\theta)^2 + (mg - P\sin\theta)^2}$

$= \sqrt{P^2\cos^2\theta + m^2g^2 - 2mgP\sin\theta + P^2\sin^2\theta}$

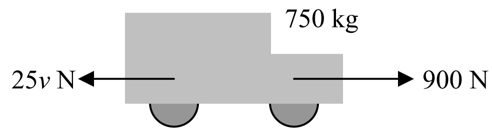
$= \sqrt{m^2g^2 - 2mgP\sin\theta + P^2}$.

(d) $\tan\phi = \frac{\mu N}{N} = \mu,\ \therefore \phi = \tan^{-1}\mu = \tan^{-1}\left(\frac{P\cos\theta}{mg - P\sin\theta}\right)$.

(e) $mg - P\sin\theta > 0$, i.e. $P < \frac{mg}{\sin\theta}$.

Example 8 A horizontal driving force of 900 N causes a car of mass 750 kg to accelerate from rest along a straight, horizontal road. The total resistance (air resistance and rolling frictions) to the car's motion is $25v$ newtons, where v ms^{-1} is the speed of the car at time t.

(a) Find the maximum speed.
(b) Set up an equation of motion of the car.
(c) Find the speed at $t = 20$ s.



(a) Maximum speed is reached when the resultant force $R = 0$.
$\therefore 900 - 25v = 0,\ v = 36\ \text{ms}^{-1}$.

(b) Equation of motion for the horizontal component:
$a = \frac{R}{m} = \frac{900 - 25v}{750} = \frac{36 - v}{30}$.

(c) Solve the differential equation $\frac{dv}{dt} = \frac{36 - v}{30}$:

$\frac{dt}{dv} = \frac{30}{36 - v},\ t = 30\int \frac{1}{36 - v}dv,\ \therefore \frac{t}{30} = -\log_e|36 - v| + C$.

At $t = 0,\ v = 0,\ \therefore C = \log_e 36,\ \therefore \frac{t}{30} = \log_e\left(\frac{36}{|36 - v|}\right)$.

Hence $\frac{36}{|36 - v|} = e^{\frac{t}{30}},\ |36 - v| = 36e^{-\frac{t}{30}},\ \therefore 36 - v = \pm 36e^{-\frac{t}{30}}$.

Since $v = 0$ at $t = 0$, $36 - v = +36e^{-\frac{t}{30}}$ is the only solution.

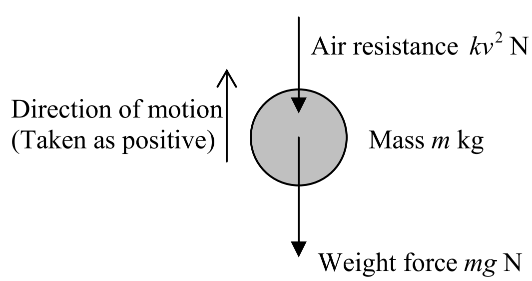
Hence $v = 36\left(1 - e^{-\frac{t}{30}}\right)$. At $t = 20,\ v = 36\left(1 - e^{-\frac{2}{3}}\right)$ ms^{-1}.

Note: As $t \to \infty,\ v \to 36$, the maximum speed found in (a).

Example 9 An object of mass m projected vertically upwards with speed u experiences air resistance kv^2, where k is a positive constant and v is the speed at time t during the upward motion.

(a) Find the maximum height above the point of projection.
(b) Find the speed of the object when it passes through the point of projection during its return journey.

(a)



Equation of motion for the vertical component:

$$a = \frac{R}{m} = \frac{-mg - kv^2}{m} = -\frac{k}{m}\left(\frac{mg}{k} + v^2\right), \text{ where } \frac{mg}{k} + v^2 > 0,$$

$$\frac{1}{2}\frac{d(v^2)}{dx} = -\frac{k}{m}\left(\frac{mg}{k} + v^2\right), \quad \frac{d(v^2)}{dx} = -\frac{2k}{m}\left(\frac{mg}{k} + v^2\right),$$

$$\frac{dx}{d(v^2)} = -\frac{m}{2k}\frac{1}{\frac{mg}{k} + v^2}, \quad x = -\frac{m}{2k}\int \frac{1}{\frac{mg}{k} + v^2} d(v^2),$$

$$\therefore -\frac{2k}{m}x = \log_e\left|\frac{mg}{k} + v^2\right| + C = \log_e\left(\frac{mg}{k} + v^2\right) + C$$

At the point of projection, $x = 0,\ v = u,\ \therefore C = -\log_e\left(\frac{mg}{k} + u^2\right)$,

$$\therefore -\frac{2k}{m}x = \log_e\left(\frac{mg}{k} + v^2\right) - \log_e\left(\frac{mg}{k} + u^2\right),$$

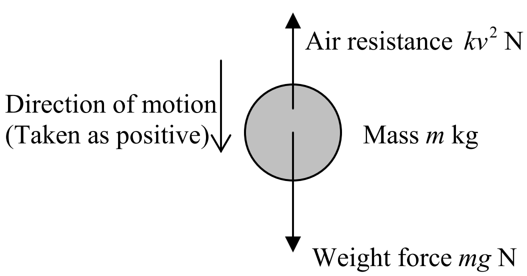
$$\therefore -\frac{2k}{m}x = \log_e\left(\frac{\frac{mg}{k} + v^2}{\frac{mg}{k} + u^2}\right).$$

Maximum height is reached when $v = 0$,

$$\therefore -\frac{2k}{m}x = \log_e\left(\frac{\frac{mg}{k}}{\frac{mg}{k} + u^2}\right), \quad x = -\frac{m}{2k}\log_e\left(\frac{1}{1 + \frac{k}{mg}u^2}\right),$$

$$\therefore x = \frac{m}{2k}\log_e\left(1 + \frac{k}{mg}u^2\right).$$

(b)



Now, take the highest point as the origin $x = 0,\ v = 0$.
Equation of motion for the vertical component:

$$a = \frac{R}{m} = \frac{mg - kv^2}{m} = \frac{k}{m}\left(\frac{mg}{k} - v^2\right), \text{ where } \frac{mg}{k} - v^2 > 0,$$

$$\frac{1}{2}\frac{d(v^2)}{dx} = \frac{k}{m}\left(\frac{mg}{k} - v^2\right), \quad \frac{d(v^2)}{dx} = \frac{2k}{m}\left(\frac{mg}{k} - v^2\right),$$

$$\frac{dx}{d(v^2)} = \frac{m}{2k}\frac{1}{\frac{mg}{k} - v^2}, \quad x = \frac{m}{2k}\int \frac{1}{\frac{mg}{k} - v^2} d(v^2),$$

$$\therefore \frac{2k}{m}x = -\log_e\left|\frac{mg}{k} - v^2\right| + C = -\log_e\left(\frac{mg}{k} - v^2\right) + C.$$

At the highest point, $x = 0,\ v = 0,\ \therefore C = \log_e\frac{mg}{k}$,

$$\therefore \frac{2k}{m}x = -\log_e\left(\frac{mg}{k} - v^2\right) + \log_e\left(\frac{mg}{k}\right),$$

$$\therefore \frac{2k}{m}x = \log_e\left(\frac{\frac{mg}{k}}{\frac{mg}{k} - v^2}\right) = \log_e\left(\frac{1}{1 - \frac{k}{mg}v^2}\right).$$

When the object returns to the point of projection,

$$x = \frac{m}{2k}\log_e\left(1 + \frac{k}{mg}u^2\right),$$

$$\therefore \log_e\left(1 + \frac{k}{mg}u^2\right) = \log_e\left(\frac{1}{1 - \frac{k}{mg}v^2}\right).$$

$$\therefore \frac{1}{1 - \frac{k}{mg}v^2} = 1 + \frac{k}{mg}u^2, \quad \therefore 1 - \frac{k}{mg}v^2 = \frac{1}{1 + \frac{k}{mg}u^2}.$$

$$\therefore \frac{k}{mg}v^2 = 1 - \frac{1}{1 + \frac{k}{mg}u^2} = \frac{1 + \frac{k}{mg}u^2 - 1}{1 + \frac{k}{mg}u^2} = \frac{\frac{k}{mg}u^2}{1 + \frac{k}{mg}u^2}.$$

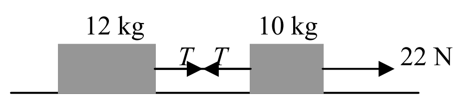
Hence $v^2 = \frac{u^2}{1 + \frac{k}{mg}u^2}, \quad \therefore v = \frac{u}{\sqrt{1 + \frac{k}{mg}u^2}}$.

Connected particles

The motion of two particles connected by a coupling is best analysed by setting up two equations of motion, one for each particle, using Newton's second law. Choose the direction of motion as the positive direction. If the first particle is pulling the second, then the second particle is pulling the first with the same force and the coupling is in tension. If the second is pushing the first, then the first is pushing the second with the same force and the coupling is in compression.

Example 1 Two boxes connected by a cord (of negligible mass) are resting on a frictionless table. The boxes have masses of 12 kg and 10 kg. A horizontal force of 22 N is applied by a person to the 10 kg box.

(a) Find the acceleration a ms^{-2} of each box.
(b) Find the tension T N in the cord.

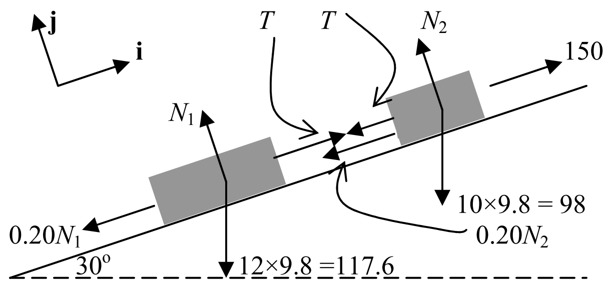


12 kg box: $T = 12a$ ....(1)
10 kg box: $22 - T = 10a$ ....(2)
Solve simultaneously, $a = 1,\ T = 12$.

Alternatively: Consider the two boxes as a single object of mass 22 kg.
$22 = 22a,\ \therefore a = 1$.
Consider the 12-kg box alone, $T = 12a = 12 \times 1 = 12$.

Example 2 Now the two connected boxes in example 1 are sliding along a plane inclined at 30^o to the horizontal. The coefficient of friction is 0.20 and the pulling force applied by the person on the 10 kg box is 150 N up and parallel to the plane.

(a) Find the acceleration a ms^{-2} of each box.
(b) Find the tension T N in the cord.



j-component:

$N_1 - 117.6\cos 30^o = 0,\ \therefore N_1 = 117.6\cos 30^o = 101.8$.

$N_2 - 98\cos 30^o = 0,\ \therefore N_2 = 98\cos 30^o = 84.9$.

i-component:

$T - 0.20N_1 - 117.6\sin 30^o = 12a$ ....(1)

$150 - T - 0.20N_2 - 98\sin 30^o = 10a$ ....(2)

Simplify (1) and (2) and then solve simultaneously:

$T - 79.16 = 12a$ ....(3)

$84.02 - T = 10a$ ....(4)

$\therefore a = 0.22$ ms^{-2}, $T = 81.81$ N.

The alternative method discussed in example 1 can be used to solve this problem.

Example 3 Suppose the cord connecting the two boxes in example 1 is placed over a frictionless pulley. Assume that both the cord and the pulley have negligible mass. The system is released from rest.

(a) Find the acceleration of each box.
(b) Find the tension in the cord.

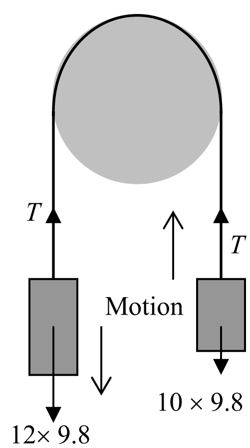
For each box take the direction of motion as the positive direction.

12-kg box: $12\times9.8 - T = 12a$ ....(1)

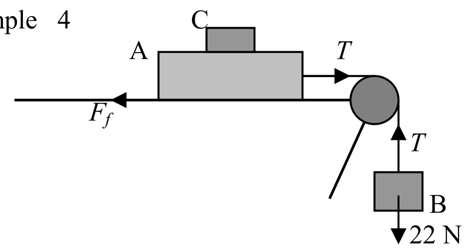
10-kg box: $T - 10\times9.8 = 10a$ ....(2)

Solve (1) and (2) simultaneously:

$a = 0.89$ ms^{-2}, $T = 106.9$ N.



Example 4



A and B are blocks with weights of 44N and 22N respectively.

(a) Determine the minimum weight (block C) that must be placed on A to keep it from sliding if the coefficient of friction between A and the table is 0.20.

(b) Block C is suddenly lifted off A. What is the acceleration of B? Assume the same coefficient of friction.

(a) Let x N be the minimum weight of C. Total weight of A and C $= (44 + x)$ N. $\therefore$ normal reaction of table $= (44 + x)$ N.

$\therefore$ at limiting equilibrium, force of friction between the table and block A, $F_f = 0.20(44 + x)$ N.

For block B: $22 - T = 0$ ....(1)

For blocks A and C: $T - 0.20(44 + x) = 0$ ....(2)

Solve (1) and (2) simultaneously: $x = 66$.

(b) Mass of block B $= \frac{22}{9.8}$ kg; mass of block A $= \frac{44}{9.8}$ kg.

For block B: $22 - T = \frac{22}{9.8}a$ ....(1)

For block A: $T - 0.20\times44 = \frac{44}{9.8}a$ ....(2)

Solve (1) and (2) simultaneously: $a = 1.96$ ms^{-2}.